On generalization of Rasiowa's implicative logics*

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In this paper (based on submitted paper [3]) we introduce two classes of logics. First, we generalize implicative logics of Rasiowa (see [9]) and get the so-called *weakly implicative logics*. Although this can be viewed as a rather minor generalization, it forces us to leave the algebraic semantics and move towards semantics provided by the so-called logical matrices (weakly implicative logics are not (in general) algebraizable in the sense of Blok and Piggozzi, see [2]).

The second defined class deals with the so called *fuzzy* logics. Many such logics were introduced and developed in the last decade of the last century. Also some well-established many-valued logics, like Łukasiewicz or Gödel-Dummett logic, have been adopted into a general framework of fuzzy logics (as extension of Hájek's Basic Fuzzy Logic BL). Since Hájek's monograph [7] the fuzzy logics are considered as *mathematical* non-classical logics *sui juris*.

It is the opinion of author, that this development has gone to the point, when we know hundreds of particular results for particular logics and so we can (and we should!) generalize them, to get results for classes of (fuzzy) logics.

The class of weakly implicative *fuzzy* logics is an attempt to *formally* define and delimit an *informal* notion of fuzzy logic. We provide the mathematical arguments demonstrating importance of this class of logics (whereas *philosophical, methodological, and pragmatical* reasons are to be find in a joint paper by the author and Libor Běhounek [1]).

1 Weakly implicative logics

We start with some basic *syntactical* definitions[1]. The notions of *propositional language* $\mathcal{L}$ and sets of *formulae* $\mathbf{FOR}_{\mathcal{L}}$ and *substitutions* are defined in the usual way. A *consecution*[2] in the language $\mathcal{L}$ is a pair $X \rhd \varphi$, where $X \subseteq \mathbf{FOR}_{\mathcal{L}}$ and $\varphi \in \mathbf{FOR}_{\mathcal{L}}$. The set of all consecutions will be denoted as $\mathcal{CON}_{\mathcal{L}}$. Since $\mathcal{CON}_{\mathcal{L}} = \mathcal{P}(\mathbf{FOR}_{\mathcal{L}}) \times \mathbf{FOR}_{\mathcal{L}}$, each subset $\mathcal{X}$ of $\mathcal{CON}_{\mathcal{L}}$ can be understood as a relation between sets of formulae and formulae (we identify the set $\mathcal{X}$ and the relation $\vdash_{\mathcal{X}}$ as: $X \vdash_{\mathcal{X}} \varphi$ iff $(X \rhd \varphi) \in \mathcal{X}$).

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[1]For the comprehensive survey into the problematic of general approach towards logical systems, consequence relations, logical matrices, etc. see [4] or [6].

[2]This term is from the Restall's book [10]. However, we use it in a very simplified version.

Logic is a structural (substitution-invariant) consequence relation in the usual sense. We understand logic as a subset of $\mathcal{CON}_{\mathcal{L}}$, i.e., the elements of a logic are consecutions.

We define the notion of (conservative) extension in the usual way. Now we introduce the notion of an axiomatic system. Observe that both axiomatic systems and logics are objects of the same kind (sets of consecutions closed under all substitutions)

Definition 1.1 (Axiomatic system) *An* axiomatic system $\mathcal{AS}$ *in language* $\mathcal{L}$ *is a non-empty set* $\mathcal{AS} \subseteq \mathcal{CON}_{\mathcal{L}}$*, which is closed under arbitrary substitution.*

The elements of $\mathcal{AS}$ *of the form* $X \rhd \varphi \in \mathcal{AS}$ *are called* axioms *for* $X = \emptyset$, n-ary deduction rules *for* $|X| = n$*, and* infinitary deduction rules *for* X *being infinite. The axiomatic system is* finitary *if all its deduction rules are finite. The axiomatic system is* pure *if its only deduction rule is modus ponens.*

Definition 1.2 (Proof) *Let* $\mathcal{AS}$ *be an axiomatic system in* $\mathcal{L}$*. An* $\mathcal{AS}$-proof *of* $T \vdash \varphi$ *(of the formula* φ *in theory* T*) is a well-founded tree labelled by formulae; the root is labelled by* φ *and leaves by either axioms or elements of* T*; and if a node is labelled by* ψ *and its preceding nodes are labelled by* $\psi_1, \psi_2, \dots$ *then* $<\{\psi_1, \psi_2, \dots\}, \psi> \in \mathcal{AS}$*. We shall write* $T \vdash^p_{\mathcal{AS}} \varphi$ *if there is a proof of* φ *in* T*.*

By theory we mean just a set of formulas. We understand the tree in an top-to-bottom way: the leaves are at the top and the root is at the bottom of the tree, so the fact that tree is well-founded just means that there is no infinitely long branch. We can show that $\vdash^p_{\mathcal{AS}}$ is the least logic containing $\mathcal{AS}$.

We say that $\mathcal{AS}$ is an *axiomatic system* for (a *presentation* of) the logic $\mathbf{L}$ iff $\mathbf{L} = \vdash^p_{\mathcal{AS}}$. Obviously, each logic has at least one presentation. We say that logic is finitary (pure) if it has some finitary (pure) presentation.

It can be shown that the logic $\mathbf{L}$ is finitary iff for each theory T and formula φ we have: if $T \vdash \varphi$ then there is finite $T' \subseteq T$ such that $T' \vdash \varphi$ (the usual definition of finitary logic). In finitary case we can linearize the tree, i.e., define the notion of the proof in the usual way.

Now we define the crucial concept of this paper: the notion of weakly implicative logic. We assume that there is a (definable) binary connective $\rightarrow$ in the propositional language. Weakly implicative logics are precisely the finitely equivalential logics with the *set of equivalence formulae* $E = \{p \rightarrow q, q \rightarrow p\}$.

Definition 1.3 (Weakly implicative logics) *Let* $\mathbf{L}$ *be a logic in* $\mathcal{L}$ *with (definable) binary connective* $\rightarrow$*. We say that* $\mathbf{L}$ *is a* weakly implicative *logic if:*

(Ref)	$\vdash_{\mathbf{L}} \varphi \rightarrow \varphi$
(MP)	$\varphi, \varphi \rightarrow \psi \vdash_{\mathbf{L}} \psi$
(WT)	$\varphi \rightarrow \psi, \psi \rightarrow \chi \vdash_{\mathbf{L}} \varphi \rightarrow \chi$
(Cng^i_c)	$\varphi \rightarrow \psi, \psi \rightarrow \varphi \vdash_{\mathbf{L}} c(\chi_1, \dots, \chi_{i-1}, \varphi, \dots, \chi_n) \rightarrow$ $c(\chi_1, \dots, \chi_{i-1}, \psi, \dots, \chi_n)$ *for each n-ary connective* c *and each* $i \leq n$*.*

The (*Ref*) is for *reflexivity*, (MP) is for *modus ponens*, (WT) is for *weak transitivity*, and (Cng) is for *congruence*. We assume neither Exchange nor Weakening nor Contraction as a rules for implication. However, we have all of them as meta-rules for $\vdash$, i.e., the connective $\rightarrow$ is by no means an internalization of $\vdash$. Thus we can say that our approach towards $\vdash$ corresponds to derivability from assumptions in some Hilbert's style calculus.

Now we introduce the notion of *linear* theory. In the existing fuzzy logic literature the term *complete* theory is usually used. However, we think that complete theory is a different concept (also known as maximal consistent theory). As this notion is crucial in our paper and we want to avoid any potential confusions we decided to pick new neutral name. The reason for choosing the name "linear" will be obvious at the end of this section.

Definition 1.4 (Linear theory) *Let* $\mathbf{L}$ *be a weakly implicative logic. A theory* T *is linear if* T *is consistent*[3] *and for each formulae* φ, ψ *we have* $T \vdash \varphi \to \psi$ *or* $T \vdash \psi \to \varphi$.

Now we recall basic *semantic* definitions. We define notions of $\mathcal{L}$-algebra and $\mathcal{L}$-matrix $\mathbf{B} = (A_{\mathbf{B}}, D_{\mathbf{B}})$ for language $\mathcal{L}$ ($A_{\mathbf{B}}$ is an algebra with the signature $\mathcal{L}$ and $D_{\mathbf{B}} \subseteq A_{\mathbf{B}}$ is a set of *designated* elements), $\mathbf{B}$-evaluation for matrix $\mathbf{B}$, and semantical consequence $\models_{\mathcal{K}}$ w.r.t. class of matrices $\mathcal{K}$. Finally, we define notion of $\mathbf{L}$-matrix ($\mathbf{L} \subseteq \models_{\{\mathbf{B}\}}$) for logic $\mathbf{L}$ and denote the class of $\mathbf{L}$-matrices by $\mathbf{MOD}(\mathbf{L})$. Now we turn our attention to a new concept:

Definition 1.5 (Matrix preorder) *Let* $\mathbf{L}$ *be a weakly implicative logic and* $\mathbf{B}$ *an* $\mathbf{L}$*-matrix. The relation* $\leq_{\mathbf{B}}$ *is defined as* $x \leq_{\mathbf{B}} y$ *iff* $x \to_{\mathbf{B}} y \in D_{\mathbf{B}}$ *is called the* matrix preorder *of* $\mathbf{B}$. *The matrix* $\mathbf{B}$ *is said to be (linearly)* ordered *iff the relation* $\leq_{\mathbf{B}}$ *is (linear) order. We denote the class of (linearly) ordered* $\mathbf{L}$*-matrixes by* $\mathbf{MOD}^*(\mathbf{L})$ *(*$\mathbf{MOD}^\ell(\mathbf{L})$ *respectively).*

Obviously for each weakly implicative logic and each $\mathbf{L}$-matrix $\mathbf{B}$ the relation $\leq_{\mathbf{B}}$ is preorder and the set designated element in $\mathbf{B}$ is an upper set w.r.t. $\leq_{\mathbf{B}}$. Furthermore if we define relation $\sim_{\mathbf{B}}$ as: $x \sim_{\mathbf{B}} y$ iff $x \leq_{\mathbf{B}} y$ and $y \leq_{\mathbf{B}} x$ we get congruence of $\mathbf{B}$. This congruence is known as Leibnitz congruence (see [6]).

Matrices for weakly implicative logics coincide with the class of the so-called *prestandard matrices* (see Dunn [5]), whereas the ordered matrices coincides with the so-called *standard matrices*. Ordered matrices also coincide with the class of the so-called *reduced matrices* in AAL (see [6]), thus we use their notation. Obviously, $\mathbf{MOD}^\ell(\mathbf{L}) \subseteq \mathbf{MOD}^*(\mathbf{L}) \subseteq \mathbf{MOD}(\mathbf{L})$.

Let us denote the weakest weakly implicative logic in the language containing only $\to$ by WIL. How WIL-matrices look like? Let $\mathbf{B}$ be a WIL-matrix, we know that $\leq_{\mathbf{B}}$ is an preorder and $D_{\mathbf{B}}$ is an upper set. Conversely, let B be a set, $\leq$ a preorder on B, and $D \neq B$ an upper set of B w.r.t. $\leq$. Let us define binary operation $\Rightarrow$ on B in *arbitrary way*, such that $x \leq y$ iff $x \Rightarrow y \in D$. Then obviously $((B, \Rightarrow), D)$ is a WIL-matrix.

We recall the concept of the Lindenbaum-Tarski matrix ($\mathbf{Lind}_T$) and show:

(1) $\mathbf{Lind}_T \in \mathbf{MOD}^*(\mathbf{L})$

(2) $\mathbf{Lind}_T \in \mathbf{MOD}^\ell(\mathbf{L})$ iff T is a linear theory.

We easily get the completeness theorem (recall that each weakly implicative logic is equivalential logic).

Theorem 1.6 (Completeness) *Let* $\mathbf{L}$ *be a weakly implicative logic. Then for each theory* T *and formula* φ *holds:* $T \vdash \varphi$ *iff* $T \models_{\mathbf{MOD}^*(\mathbf{L})} \varphi$.

[3] I.e., there is φ such that $T \nvdash \varphi$

2 Weakly implicative fuzzy logics

In the previous section we have seen that each weakly implicative logic is sound and complete w.r.t. the class of its *ordered* matrices. There is an obvious question, which of them are complete w.r.t. class of its *linearly ordered* matrices. This will lead us to the second central definition of this paper: the notion of weakly implicative fuzzy logics.

Definition 2.1 *A weakly implicative logic* $\mathbf{L}$ *is* fuzzy *if* $\mathbf{L} = \models_{\mathbf{MOD}^{\ell}(\mathbf{L})}$.

The full proper name of the above-defined class of logics is *weakly implicative fuzzy logics*. However from now on we use the term (fuzzy) logic instead of weakly implicative (fuzzy) logic.

Definition 2.2 *Let* $\mathbf{L}$ *be a logic. We say that* $\mathbf{L}$ *has:*

- *the* Linear Extension Property (LEP) *if for each theory* T *and formula* φ *such that* $T \nvdash \varphi$ *there is a linear theory* $T' \supseteq T$ *and* $T' \nvdash \varphi$.
- *the* Prelinearity Property (PP) *if for each theory* T *we get* $T \vdash \chi$ *whenever* $T, \varphi \to \psi \vdash \chi$ *and* $T, \psi \to \varphi \vdash \chi$.
- *the* Subdirect Decomposition Property (SDP) *if each ordered* $\mathbf{L}$*-matrix is a subdirect product*[4] *of linear* $\mathbf{L}$*-matrices.*

We can show that each logic with LEP has also PP. To reverse this claim we need one additional assumption: the logic has to be finitary[5]. We present the main theorem of this section:

Theorem 2.3 *A logic* $\mathbf{L}$ *is a fuzzy iff it has* LEP*. Furthermore, if* $\mathbf{L}$ *is finitary then the following are equivalent:*

(1) $\mathbf{L}$ *is a fuzzy logic*

(2) $\mathbf{L}$ *has* LEP

(3) $\mathbf{L}$ *has* PP

(4) $\mathbf{L}$ *has* SDP.

If we define $\varphi^0 \to \psi = \psi$ and $\varphi^{i+1} \to \psi = \varphi \to (\varphi^i \to \psi)$, we can prove that in each fuzzy logic we have $(\varphi \to \psi)^i \to \chi, (\psi \to \varphi)^j \to \chi \vdash_{\mathbf{L}} \chi$ for each natural i and j. Furthermore, we can show that:

- The intersection of an arbitrary system of fuzzy logics is a fuzzy logic.
- An axiomatic extension of arbitrary fuzzy logic is a fuzzy logic.
- Let $\mathbf{L}'$ be a fuzzy logic, which is a conservative expansion of a finitary logic $\mathbf{L}$. Then $\mathbf{L}$ is fuzzy logic.

Thus we can soundly define:

Definition 2.4 *Let* Q *be a set of consecutions and* $\mathbf{L}$ *a logic. We denote the weakest fuzzy logic with* $Q \subseteq \mathbf{L}$ *as* $\mathcal{FUZZ}(Q)$.

[4] Defined in the usual way, see [4]

[5] However it is quite obvious that for some infinitary rules the equivalence holds as well

3 Extensions of BCI

Let us recall that BCI is an implicational fragment of intuitionistic linear logic, it is not implicative logic, and it has the following presentation:

$\mathcal{B}$	$\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
$\mathcal{C}$	$\vdash (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
$\mathcal{I}$	$\vdash \varphi \to \varphi$
(MP)	$\varphi, \varphi \to \psi \vdash \psi$.

Now we present an important definition—the deduction theorem—for rather wide class of weakly implicative logic. We present it in an unusual form. Our formulation allows us to show exactly which logics have this deduction theorem.

Definition 3.1 *Let* **L** *be a logic. We say that* **L** *has* Implicational Deduction Theorem *(*$\mathrm{DT}_{\to}$*) if* **L** *has a* pure *presentation* $\mathcal{AX}$ *and for each theory* T*, formulae* φ, ψ*, and for each* $\mathcal{AX}$*-proof* $\mathcal{P}$ *of* $T, \varphi \vdash \psi$ *there is an* $\mathcal{AX}$*-proof* $\mathcal{P}'$ *of* $T \vdash \varphi^n \to \psi$*, where* n *is a number of occurrences of* φ *in the leaves of the proof* $\mathcal{P}$ *and each* $\chi \in T$ *occurs in the leaves of* $\mathcal{P}$ *the same number of times as in the leaves of* $\mathcal{P}'$*.*

Theorem 3.2 (Deduction theorem) *Let* **L** *be a logic. Then* **L** *has* $\mathrm{DT}_{\to}$ *iff* **L** *is pure and* **L** *is an expansion of* BCI*.*

Corollary 3.3 *Let* **L** *be a finitary logic expanding* BCI*. Then the following are equivalent:*

- **L** *has* $\mathrm{DT}_{\to}$
- **L** *has* LDT *(Local Deduction Theorem: for each theory* T *and formulae* φ, ψ*:* $T, \varphi \vdash \psi$ *iff there is* n *such that* $T \vdash \varphi^n \to \psi$*.)*
- **L** *is* pure*.*

In the presence of LDT we can get an equivalent definition of fuzzy logics:

Lemma 3.4 *Let* **L** *be a finitary logic with* LDT*. Then* **L** *is a fuzzy logic iff for each* i *and* j *holds:* $(\varphi \to \psi)^i \to \chi, (\psi \to \varphi)^j \to \chi \vdash_{\mathbf{L}} \chi$ *.*

At the end of this section we present an infinite axiomatic system for the weakest fuzzy logic extending BCK. The question whether there is finite system seems to be open. Recall that logic BCK is axiomatized by adding axiom $\mathcal{K}$ to the axioms of BCI.

Definition 3.5 *The* fuzzy BCK logic *(*FBCK*) is an extension of* BCI *by:*

$\mathcal{K}$	$\vdash \varphi \to (\psi \to \varphi)$	
F_n	$\vdash ((\varphi \to \psi)^n \to \chi) \to (((\psi \to \varphi)^n \to \chi) \to \chi)$	*for all* n

Theorem 3.6 $\mathrm{FBCK} = \mathcal{FUZZ}(\{\mathcal{B}, \mathcal{C}, \mathcal{K}\})$.

We can alter the axioms F_n by using two different natural numbers m, n as "exponents", we get axioms $\mathrm{F}_{m,n}$. If we add axioms $\mathrm{F}_{m,n}$ to the BCI logic, we get fuzzy logic (by Lemma 3.4). However, we are not able to the prove the converse statement, i.e., that this logic is the weakest fuzzy logic over BCI.

Corollary 3.7 *The subquasivariety of* BCK*-algebras generated by the class of* BCK*-chains is axiomatized by adding the identities*
$((x \to y)^n \to z) \to (((y \to x)^n \to z) \to z) = 1$ *for each* n*.*

This result is achieved in the paper by Olmedo and Salas [8], where the authors propose a concept of "linearization" of a logical systems (of an algebraizable logic). Their approach is rather different from ours in several aspects. They take an algebraizable logic and the quasivariety $\mathcal{Q}$ of algebras constituting the equivalent algebraic semantics for the original logic. Then they define a logic given by a quasivariety generated by chains from $\mathcal{Q}$. Their paper is utilizing a rather heavy algebraic machinery to get the result, whereas we get it as a rather simple consequence of a deduction theorem.

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