ALGEBRAIC TREATMENT OF INFINITISTIC DEFINITIONS

JANUSZ CZELAKOWSKI

In this paper a uniform and general theory of definition, encompassing infinitistic, semantic methods of defining abstract objects, is outlined. The focus of the paper is on three basic infinitistic ways of defining objects:

(1) the method of fixed-points,
(2) the method of algebraic completions of posets,
(3) the induction method.

The development of the deductive sciences, and of mathematics in particular, shows the growing, significant role of various methods of defining abstract objects. The theory of definition together with the widely understood proof theory provide basic, logical tools in these sciences. It is rather a common opinion that the theory of definition is a less developed, as compared with proof theory, part of logic. While the proof-theoretic methods, either based on the Hilbertian notion of a proof, or built on Gentzen formalizations linked with the so called natural deduction, form a well-developed, fully fledged branch of logic, the theory of definition is not so positively assessed. It is difficult to say what is the reason for this situation. The theory of definition isolates a number of plausible types of defining abstract objects. This theory is also concerned with the categorization of various definitions and ways of introducing them into the language. In the standard textbooks of logic or the theory of models, one discriminates between two forms of definability: *explicit* definability and *implicit* definability. The first type of defininig basically reduces to the art of making plausible, handy linguistic abbreviations within a given language. Definitions *explicite* thus correspond to the traditional nominal definitions. In turn, definitions *implicite* form a more subtle semantic tool. In light of Beth’s Theorem the two methods, when restricted to first-order languages, are basically equivalent. These methods are finitary in the sense that they make no reference, either directly or indirectly, to the notion of infinity (potential or actual). These methods do not exhaust the arsenal of available types of definitions one may encounter in the literature. The important feature of many definitions is that they are inherently *inifitistic.* Definitions of this type make reference to infinite methods and constructions either by applying to the notion of a limit or by making use of properties of certain infinite sets or classes as e.g. the classes of ordinal numbers or well-founded relations. The definitions of the limit of a sequence, the limit of a function at a point, the derivative of a function or the Riemann integral of a function are examples of such definitions. Chronologically, the method of exhaustion, invented by the Greeks, is the first sample of such definitions.

The treatment of infinitistic definitions based on the algebraic completions of posets seems to be not widely recognized in the literature (though the concept of an algebraic poset, relevant to that method, is due to Dana Scott). It is characterized by the following features:

(1) This is a *uniform* treatment, enabling one to derive the variety of infinitistic definitions from a few, simple principles.
(2) This is a *general* treatment, defining a certain general scheme of infinitistic definitions.
(3) This is a *qualitative* treatment, not refering to known numerical or analytic methods as primitive principles, but rather based on simple notions borrowed from the theory of partial order.

The notions of a *directed-complete partially ordered set* (a *directed-complete poset*, for short) and of the *algebraic completion* of a poset play a key role in this approach. A poset Q is *directed-complete* if every directed subset has a supremum in Q. The second notion is a straightforward generalization of the concept of an algebraic lattice, well-known in many semantic contexts. Every poset P possesses a unique, up to isomorphism, algebraic completion in which P coincides with the set of compact elements of this completion. Moreover, every monotone mapping from P into an arbitrary directed-complete poset Q has a unique order-continuous extension from the algebraic completion of P to Q. This extension is defined in a certain canonical way.

The above, simple observations define a certain general scheme of infinitistic definitions. This scheme is briefly defined as follows. At the outset there is a directed poset P_0 and a monotone mapping F_0 defined on P_0 with values in a directed-complete poset Q. The elements of the image $F_0[P_0]$ are called *approximations to the defined concept*. As P_0 is directed, the unique algebraic completion P of P_0 has the greatest element **1**. After the identification of the poset P_0 with the set of compact elements of the completion P, the mapping F_0 uniquely extends to an order continuous mapping F from P to Q. Then the limit value F(**1**), belonging to Q, defines the infinitistic object approximated by the values $F_0[P_0]$. The above scheme is thus anchored in the theory of order and does not refer to primitive concepts having a distinct quantitative character like numbers or sets of numbers. This scheme encompasses the definitions rooted in calculus making use of the notion of a limit as e.g. the limit of a sequence, the Riemann integral or the Jordan measure. In the paper we thoroughly examine the scope of this method by providing a list of useful applications and instances of the method.

An interesting feature of the above approach to infinitistic definitions is the conceptul simplicity of its intended models, i.e., structures it deals with. These are posets and their algebraic completions. But there is yet another type of infinitistic definitions and another interesting aspect of this approach. Its models are strongly linked with underlying structures of the theory of fixed-points. For several years the methods based on the so called semantics of fixed points have been extensively developed. This sort of semantics applies a range of results, conventionally called fixed-point theorems, to defining or constructing abstract objects with given prescribed properties. Fixed-point theorems enable one to prove the existence of this objects, and often – to show their uniqueness. This part of semantics may be treated as a non-standard branch of the theory of definition. The methods of defining the semantics of fixed points offeres are inherently infintistic. The heart of the matter is in defining abstract things as fixed points, and often – as least fixed points, of certain mappings of a partially ordered set into itself. In the simplest case, the defined object is approximated by certain constructions performed in a finite number of steps. By passing to the limit (in the order sense) one obtains the object with the required properties. This process takes plays in ordered structures in which the passage to the infinite is allowed and the existence of the limit secured. The fixed-point approach to semantics is therefore meaningful only for ordered structures which exhibit some completeness properties as e.g. the existence of suprema for directed subsets or chains. The completeness of a poset guarantees that among its elements there always exists at least one for which the above infinite process terminates. The process of definability by means of fixed-points methods thus consists in picking out from the set of pre-existed objects, forming a complete poset, the one with the desired properties.

Fixed-point theorems form the foundation of a part of theoretical computer science called the theory of *semantic domains*. This theory is developed mainly by Dana Scott and his collaborators. In this work we present new results concerning fixed-points and show applications of these results in various disciplines. We are mainly concerned with fixed-point theorems for *relations* defined in posets and not only for mappings. (We note that traditionally the focus of mathematical literature is on fixed-point theorems for *functions*, and not relations.) We present here a number of such theorems and also show their applicability in such diverse disciplines like model theory, universal algebra, functional analysis, and the theory of differential equations. We advocate the thesis that despite conceptual differences between these branches of mathematics, a significant part of their proof theory is essentially based on similar methods referring either to fixed-points constructions for relations for the method of algebraic completions of posets. E.g. the Lower Löwenheim-Skolem Theorem is a simple corollary to a certain fixed-point theorem for relations. Many canonical constructions of models (in the sense of the theory of models, see Chang and Keisler [1990]) are also obtained as fixed-points for relations. On the other hand, the well-known Pickard-Lindelöf Theorem on the existence and uniqueness of a solution of the Cauchy problem in the theory of defferential equations can be proved by means of applying some fixed-point theorem.

The method of fixed-points blurs (to some extent) the differences between proof theory and semantics. According to this method, the proof a theorem, especially of an existential one, consists in the construction of an appropriate poset (with some completeness property) along with a suitable binary relation defined on this poset, and then showing that a fixed-point of that relation has the properties of the object whose existence one wishes to show.

On the other hand, the development of set theory gave a new stimulus to the theory of definition. Set theory is percived as an ontology of mathematics. All conceivable mathematical constructions are created in accordance with the principles provided by set theory, and its axioms in particular. The process of incorporating new concepts into mathematics takes place within set theory – the new concepts are defined (or built, as one wishes) by means of applying the methods borrowed from the arsenal provided by set theory.The Principle of Definability by Transfinite Recursion as well as its variants occupy a distinctive place in mathematics. Definability by means of Transfinite Recursion is nowadays the standard, or even routine, technique of introducing new concepts to mathematics and the formal semantics. The Principle of Definability by Transfinite Recursion is deducible from the Principle of Transfinite Induction. Set theory offers a number of induction principles which are useful in the foundations of mathematics as e.g. the Principle of Induction for Regular sets, widely applied in the theory of Boolean extensions of models of set theory. In Czelakowski's papers ”Induction Principles for sets” and ”Set-theoretic domains” a systematic, general account of various set-theoretic induction principles is presented. These principles are the key which makes it possible to ”algebraize” set theory. The structures, being the result of the process of algebraization, are called set-theoretic domains. These are certain posets (or even ordered classes) in which subsets bounded from above possess supremums. Each such a poset is equipped with a binary operation which, roughly speaking, encodes the properties of the epsilon relation. Furthermore, each domain has a built-in a certain plausible induction scheme. Set-theoretic domains are usually not first-order definable but they are, to a large extent, coextensive with the „usual” models of set theory. It is important to note that many of the axioms of set theory as e.g. the Axiom of Choice, the Axiom of Regularity etc. can be shown to be transformed into equivalent induction principles. Each such induction principle has built-in an ”infinitistic” factor. In the simplest case, this infinitistic step is formulated (for sets) as follows:

> Suppose *P* is a property of sets. Let *D* be a directed family of sets. If *P* holds for every set *X* belonging to *D*, then *P* holds for the union $\cup D$.

The induction principle with the above component (and where *P* is elementary i.e., it is defined by a formula of set theory) is provable in ZF. This principle distinguishes set-theoretic domains equivalent to models of ZF. There are also induction principles in which *D* is assumed to be a chain or a well-ordered chain.

The above papers provide a description and classification of induction principles for sets and formulate various conditions, equivalent to these principles, but having more direct set-theoretic connotations.

Many of the above induction principles determine definite methods of defining various objects in the associated set-theoretic domains in much the same way as, say, the Principle of Transfinite Induction determines the Principle of Definability by Transfinite Recursion. These issues are discussed more thoroughly in „Set theoretic domains”.

We argue in the paper that the method of fixed-points is, in many cases, equivalent to the induction method. Modulo some weak set-theoretic assumptions (weaker than ZF), each of fixed-point theorem for relations is either equivalent to the Axiom of Choice or to the well-known Principle of Definabilty by Transfinite Recursion or it is equivalent to the usual arithmetic Principle of Definability by Recursion. The significance of fixed-point theorems lies rather in their elegance, brevity and adaptability to various situations in which one prefers to operate directly rather with ordered structures thasn natural or ordinal numbers.

4.1. Partially Ordered Sets.

Let P be a set. A binary relation $\leq$ on P is an *order* (or *partial order*) on P iff $\leq$ satisfies the following conditions:

(i) $\leq$ is *reflexive*, i.e., $a \leq a$, for all $a \in A$;

(ii) $\leq$ is *transitive*, i.e., $a \leq b$ and $b \leq c$ implies $a \leq c$, for all a, b, $c \in A$;

(iii) $\leq$ is *antisymmetric*, i.e., $a \leq b$ and $b \leq a$ implies $a \leq b$, for all a, $b \in A$.

A *partially ordered set*, a *poset*, for short, is a set with an order defined on it.

Each order relation $\leq$ on P gives rise to a relation $<$ of *strict order* : $a < b$ in P iff $a \leq b$ and $a \neq b$.

Let $(P, \leq)$ be a poset and let X be a subset of P. Then X inherits an order relation from P : given $x, y \in X$, $x \leq y$ in X iff $x \leq y$ in P. We then also say that the order on X is *induced* by the order from P.

(1) An element $M \in X$ is called *maximal* in X whenever $M \leq x$ implies $M = x$, for every $x \in X$.

(2) An element $m \in X$ is called *minimal* in X whenever $x \leq m$ implies $m = x$, for every $x \in X$.

(3) An element $u \in P$ is called *an upper bound* of the set X if $x \leq u$, for every $x \in X$.

(4) An element $l \in P$ is called a *lower bound* of the set X if $l \leq x$, for every $x \in X$.

(5) An element $a \in P$ is called the *least upper bound* of the set X if a is an upper bound of X and $a \leq u$ for every upper bound u of X. If X has a least upper bound, this is called the *supremum* of X and is written ”sup(X)”.

(6) An element $b \in P$ is called the *greatest lower bound* of the set X if b is a lower bound of X and $l \leq b$ for every lower bound l of X. If X has a greatest lower bound, this is called the *infimum* of X and is written ”inf(X)” .

We note that X may have more than one maximal element, or none at all. A similar situation holds for minimal elements.

Instead of ”sup(X)” and ”inf(X)” we shall often write ”$\vee X$ ” and ”$\wedge X$ ” ; in particular we write ”$a \vee b$ ” and ”$a \wedge b$ ” instead of ”sup($\{a, b\}$) ” and ”inf($\{a, b\}$) ”.

If the poset P itself has an upper bound u, then it is the only uper bound. u is then called the *greatest element* of P. In an analogous way the notion of the *least element* of P is defined.

A set $X \subseteq P$ is :

(a) an *upper directed* subset of P if for every pair a, $b \in X$ there exists an element $c \in X$ such that $a \leq c$ and $b \leq c$ (or, equivalently, if every finite non-empty subset of X has an upper bound which is an element of X);

(b) a *chain* in P if, for every pair a, $b \in X$, either $a \leq b$ or $b \leq a$ (that is, if any two elements of X are comparable)

(c) a *well-ordered subset* of P (or: a *well-ordered chain* in P) if X is a chain in which every non-empty subset $Y \subseteq X$ has a minimal element (in Y). Equivalently, X is well-ordered iff it is a chain and there is no strictly decreasing sequence $c_0 > c_1 > ... > c_n > ...$ of elements of X. Every well-ordered is isomorphic with a unique ordinal, called the *type* of X.

A subset directed downwards is defined similarly; when nothing to the contrary is said, "directed" will always mean "directed upwards". If the poset $(P, \leq)$ itself is a chain or directed, then it is simply called a chain or a directed poset.

Let $(P, \leq)$ be a poset and let X be a subset of P. The set X is *cofinal in* $(P, \leq)$ if for every $a \in P$ there exists $b \in X$ such that $a \leq b$. If X is cofinal in $(P, \leq)$, then $\sup(P)$ exists iff $\sup(X)$ exists. Furthermore $\sup(P) = \sup(X)$.

Theorem 4.1.1. *Let* $(P, \leq)$ *be a poset. Every countable directed subset D of* $(P, \leq)$ *contains a well-ordered subset of type* $\leq \omega$. *In particular, every countably infinite chain contains a cofinite well-ordered subchain of type* ω. □

The above theorem is not true for uncountable directed subsets.

Theorem 4.1.2. (Zorn's Lemma). *If every non-empty chain in a poset P has an upper bound, then the set P contains a maximal element.* □

Zorn's Lemma (quantified over all posets) is an equivalent form of the Axiom of Choice (on the basis of the familiar axioms of Zermelo-Fraenkel's set theory ZF without the Axiom of Regularity).

Definition 4.1.3. Let $(P, \leq)$ be a poset.

(1) The poset $(P, \leq)$ is *directed-complete* if for every directed subset $D \subseteq P$, the supremum $\sup(D)$ exists in $(P, \leq)$.

(2) The poset $(P, \leq)$ is *chain-complete* (or *inductive*) if for every chain $C \subseteq P$, the supremum $\sup(C)$ exists in $(P, \leq)$.

(3) The poset $(P, \leq)$ is *well-ordered chain-complete* if for every well-ordered chain $C \subseteq P$, the supremum $\sup(C)$ exists in $(P, \leq)$. □

It is clear that every directed-complete poset is chain-complete and every inductive poset is well-ordered chain-complete. The above properties are thus succesively weaker and weaker. It tuns out however that in the presence of the Axiom of Choice they are mutually equivalent (see Appendix).

The empty subset of a poset is well-ordered. Hence, if (P, $\leq$) is well-ordered chain-complete, then the supremum of the empty subset exists and it is the least element in (P, $\leq$). This element is often denoted by **0** and called the *zero* of the poset P. Thus every well-ordered chain-complete poset has zero. It follows that in every inductive poset and in every directed-complete poset the least element exists.

Let (P, $\leq$) be a poset. A mapping $\pi : P \to P$ is *monotone* if $a \leq b$ implies $\pi(a) \leq \pi(b)$ for every pair a, $b \in A$.

It is easy to see that if π is monotone and C is a chain (a well-ordered chain) in (P, $\leq$), then the image $\pi[C\}$:= $\{\pi(a) : a \in C\}$ is also a chain (a well-ordered chain). A similar implication holds for every directed set $D \subseteq P$.

4.2. Fixed-Point Theorems for Relations.

Let $R \subseteq P \times P$ be a binary relation defined on a non-empty set P. An element $a^* \in P$ is called a *fixed-point* of R if $a^* \, R \, a^*$ holds.

We begin with the following two simple observations. A relation $R \subseteq P \times P$ is called *serial* if it satisfies the condition: for every $a \in P$ there exists an element $b \in P$ (not necessarily unique) such that $a \, R \, b$ holds, symbolically

(1) $(\forall a \in P \, \exists b \in P) \; a \, R \, b$.

Let (P, $\leq$) be a poset. A relation $R \subseteq P \times P$ is called $\forall$-*expansive* if it is serial and included in $\leq$, i.e.,

(2) $(\forall a, b \in P)$ *$a \, R \, b$ implies $a \leq b$*.

Theorem 4.2.1. (The Fixed-Point Theorem for $\forall$-Expansive Relations). *Let (P, $\leq$) be a poset in which every non-empty chain has an upper bound. Let $R \subseteq P \times P$ be a $\forall$-expansive relation. Then R has a fixed-point a^* which additionally satisfies the following condition*:

(3) *for every $b \in P$, if $a^* \, R \, b$, then $b = a^*$.*

(This means that a^* is the unique element b in P such that $a^* \, R \, b$, i.e., $\{b \in P : a^* \, R \, b\} = \{a^*\}$.)

.

Proof. By Zorn's Lemma, applied to (P, $\leq$), there exists at least one maximal element a^* (in the sense of $\leq$). We show that a^* is a fixed-point for R. By seriality, there exists $b \in P$ such that $a^* \, R \, b$. (2) then implies that $a^* \leq b$. Since a^* is maximal, we have that $a^* = b$. Hence $a^* \, R \, a^*$. So a^* is a fixed-point of R. Evidently, by maximality and (2), a^* also satisfies (3). □

It is obvious that if R is a function on P, that is, R satisfies the condition: for every $a \in P$ there exists a unique element $b \in P$ such that $a\,R\,b$ holds, symbolically:

$$(\forall a \in P\ \exists! b \in P)\ a\,R\,b,$$

then every fixed-point of R satisfies (3).

Let $(P, \leq)$ be a poset. A relation $R \subseteq P \times P$ is called ∃-*expansive* if for every $a \in P$ there exists $b \in P$ such that $a\,R\,b$ and $a \leq b$, symbolically:

(4) $(\forall a \in P\ \exists b \in P)\ a\,R\,b$ *and* $a \leq b$.

Evidently, every ∃-expansive relation is serial and every ∀-expansive relation is ∃-expansive. It is clear that if R is a function from P to P, then the properties of being ∀-expansive and ∃-expansive are equivalent for R.

Theorem 4.2.2. (The Fixed-Point Theorem for ∃-Expansive Relations). *Let* $(P, \leq)$ *be a poset in which every non-empty chain has an upper bound. Let* $R \subseteq P \times P$ *be a* ∃*-expansive relation. Then* R *has a fixed-point* a^* *which additionally satisfies the condition* :

(5) $(\forall b \in P)\ a^*\,R\,b$ *and* $a^* \leq b$ *implies* $b = a^*$.

Proof. Define the relation $R_0 := R \cap \leq$. The relation R_0 is serial and ∀-expansive. By Theorem 4.2.1, R_0 has a fixed-point a^* for which (3) holds. Consequently, $a^*\,R\,a^*$ and (5) readily follows. □

The proof of Theorem 4.2.1 employs the Axiom of Choice (in the form of Zorn's Lemma). But in fact, the set-theoretic status of Theorems 4.2.1 and 4.2.2 is the same - each of the above fixed-point theorems is equivalent to the Axiom of Choice.

Theorem 4.2.3. *On the basis of Zermelo-Fraenkel set theory* ZF (*without the Axiom of Regularity*), *the following conditions are equivalent* :

(a) *The Axiom of Choice* (AC).
(b) *Theorem* 4.2.1.
(c) *Theorem* 4.2.2.

Proof. The implication (a) ⇒ (b) directly follows from the proof of Theorem 4.2.1 because Zorn's Lemma is used here. The implication (b) ⇒ (c) is present in the proof of Theorem 4.2.2. To prove the implication (c) ⇒ (a), assume that Theorem 4.2.2 holds. We show that then Zorn's Lemma holds. For let $(P, \leq)$ be an arbitrary poset in which every non-empty chain has an upper bound. Let $R := \leq$. The relation R is evidently ∃-expansive. By Theorem 4.2.2, R has a fixed-point a^* which satisfies (5). We show a^* is a maximal element in $(P, \leq)$. Assume $b \in P$ and $a^* \leq b$. So $a^*\,R\,b$ and $a^* \leq b$, by the definition of R. Condition (4) then gives that $b = a^*$. This means that a^* is a maximal element in $(P, \leq)$. □

Theorems 4.2.1 and 4.2.2 are possibly strongest fixed-point theorems one can encounter in the literature because proofs of any results on fixed-points comparable in strength and generality with the above theorems would require set-theoretic assumptions at least as strong as the Axiom of Choice.

Example. Let $P = [0,1]$ be the closed unit interval of real numbers. Evidently, the system $(P, \leq)$ with the usual ordering $\leq$ of real numbers satisfies the hypothesis of Theorems 4.2.1 and 4.2.2. If the relation R is taken to be equal to $\leq$ on P , then R is $\forall$-expansive. Hence it has a fixed-point in $(P, \leq)$ which additionally satisfies (3). It is clear that 1 is the only such a fixed-point of R. On the other hand, every element of P is a fixed-point of R. □

Let $(P, \leq)$ be a poset. A mapping $\pi : P \to P$ is *expansive* if $a \leq \pi(a)$ for every $a \in A$.

Corollary 4.2.4. (Zermelo). *Let* $(P, \leq)$ *be a poset in which every non-empty chain has an upper bound. Let* $\pi : P \to P$ *be an expansive mapping. Then* π *has a fixed- point*, *i.e.*, *there exists* $a^* \in P$ *such that* $\pi(a^*) = a^*$. *Furthermore,* a^* *can be assumed to be a maximal element in* $(P, \leq)$.

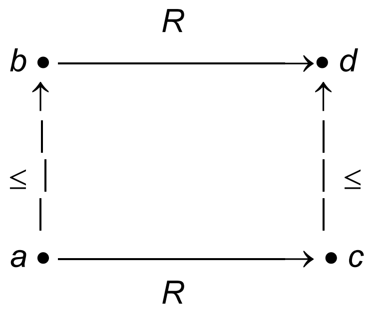
Proof. Following set theory, each function is identified with its graph. Accordingly, let R_π be the graph of π. Thus $a\, R_\pi\, b$ holds iff $b = \pi(a)$, for all $a, b \in P$. The relation R_π satisfies the hypothesis of both Theorems 4.2.1 and 4.2.2. R_π is serial because π is a function. R_π is $\forall$-expansive because it is serial and π is expansive. In virtue of Theorem 4.2.1, R_π has a fixed-point a^*. In particular, $a^*\, R_\pi\, a^*$ holds which means that $a^* = \pi(a^*)$. □

The above theorems prove the existence of rather big fixed-points – they were maximal elements in a given poset. The theorems we present farther are concerned with the problem of finding possibly small fixed-points.

Definition 4.2.5. Let $(P, \leq)$ be a poset. A binary relation $R \subseteq P \times P$ is called *monotone* if it satisfies:

$$(\forall\, a,b,c \in P)[\, a \leq b \text{ and } a\, R\, c \text{ implies } (\exists\, d \in P)\, b\, R\, d \text{ and } c \leq d\,]$$

(see the figure below). □



In diagrams like this, horizontal arrows are labeled by R and the vertical arrows are labeled by the order sign $\leq$.

If $\pi : P \to P$ is a function, then π is monotone iff the graph R_π of π is a monotone relation in the above sense.

A poset (P, ≤) is called *chain- σ-complete* (or *linearly- σ-complete*) if every countable chain in (P, ≤) has a supremum.

In the sequel we are concerned with σ-continuous relations. Various non-equivalent definitions of σ-continuity are admissible here. Here we work with the following one.

Definition 4.2.6. Let (P, ≤) be a chain-σ-complete poset and let $R \subseteq P \times P$ be a binary relation.

1. R is called *chain- σ-continuous* if R is monotone and it additionally satisfies the following condition:

$(\text{cont})_\sigma$ For every non-empty chain C in (P, ≤) of type ω and for every monotone mapping $f : C \to P$, if $a \; R \; f(a)$ for all $a \in C$, then $\sup(C) \; R \; \sup(f[C])$.

2. R is *chain- σ-continuous in the stronger sense* if it is chain- σ-continuous and additionaly satisfies the following condition:

$(*)_\sigma$ For every chain D in (P, ≤) of type ω and for every element $a \in P$, if $a \; R \; d$ for all $d \in D$, then $a \; R \; \sup(D)$. □

We note that $(\text{cont})_\sigma$ can be equivalently formulated as follows:

For every strictly increasing sequence of elements of P,

$$a_0 < a_1 < \ldots < a_n < a_{n+1} < \ldots$$

and every increasing sequence

$$b_0 \leq b_1 \leq \ldots \leq b_n \leq b_{n+1} \leq \ldots$$

of elements of P, if $a_n \; R \; b_n$ for all n, then $\sup\{a_n : n \in \omega\} \; R \; \sup\{b_n : n \in \omega\}$.

It is not difficult to prove that R is chain-σ-continuous in the stronger sense iff it is monotone and satisfies:

$(\text{cont})^*_\sigma$ For any two monotone mappings $f : \omega \to P$ and $g : \omega \to P$, if $f(n) \; R \; g(n)$ for all $n \in \omega$, then $\sup(f[\omega]) \; R \; \sup(g[\omega])$.

It is easy to see that in the light of Theorem 4.1.1, in the condition $(\text{count})_\sigma$ we may equivalently quantify over arbitrary countably infinite chains. The same remark applies to the condition $(*)_\sigma$.

We observe that the antecedent of $(\text{count})_\sigma$ is not vacuously satisfied for certain relations. We have:

Observation. *Let* $(P, \leq)$ *be a poset. If a relation* $R \subseteq P \times P$ *is a monotone and serial , then for every non-empty countably infinite chain* $a_0 < a_1 < ... a_n < a_{n+1} < ...$ *in* $(P, \leq)$, *there exists a chain* $b_0 \leq b_1 \leq ... \leq b_n \leq b_{n+1} \leq ...$ *in* $(P, \leq)$ *such that* $a_n R b_n$ *for all n*.

Indeed, let $a_0 < a_1 < ... a_n < a_{n+1} < ...$ be a countable chain. As R is serial, there exists an element $b_0 \in P$ such that if $a_0 R b_0$. As $a_0 \leq a_1$ and $a_0 R b_0$, the monotonicity of R implies the existence of an element $b_1 \in P$ such that $a_1 R b_1$ and $b_0 \leq b_1$. As $a_1 \leq a_2$ and $a_1 R b_1$, there exists an element $b_2 \in P$ such that $a_2 R b_2$ and $b_1 \leq b_2$, again by monotonicity. Going farther, as $a_2 \leq a_3$ and $a_2 R b_2$, there exists an element $b_3 \in P$ such that $a_3 R b_3$ and $b_2 \leq b_3$. Continuing this procedure, we define a countable chain $b_0 \leq b_1 \leq ... \leq b_n \leq b_{n+1} \leq ...$ *in* $(P, \leq)$ such that $a_n R b_n$ for all n. □

Given a binary relation R on a set P and $a \in P$, we define:

$$R[a] := \{b \in P : a R b\}.$$

$R[a]$ is called the *R-image* of the element a.

Given a poset $(P, \leq)$, a binary relation R on P and $a \in P$, define:

$$[a) := \{b \in P : a \leq b\},$$

$$(a] := \{b \in P : b \leq a\}.$$

The basic observation concerning fixed-points of σ-continuous relations is provided by the following theorem:

Theorem 4.2.7. *Let* $(P, \leq)$ *be a σ-chain-complete poset. Every chain- σ-continuous relation* $R \subseteq P \times P$ *such that the set* R **[0]** *is non-empty has a fixed-point* a^*. *Furthermore* a^* *can be assumed to have the following property*: *for every* $y \in P$, *if* $R[y] \subseteq (y]$, *then* $a^* \leq y$.

Proof. **0** stands for the least element in $(P, \leq)$. **0** is the supremum of the empty chain. As R is chain-σ-continuous, R is a monotone relation in $(P, \leq)$.

We define:

$$Q := \{x \in P : R[x] \cap [x) \neq \emptyset \text{ and } (\forall y \in P)\,(R[y] \subseteq (y] \text{ implies } x \leq y)\}.$$

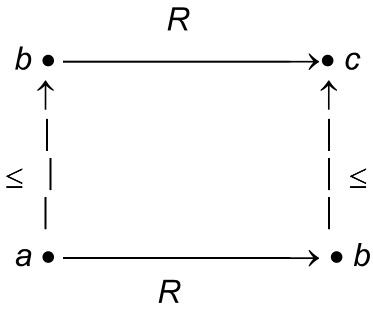
Evidently $\mathbf{0} \in Q$. Hence the set Q is non-empty. We begin with the following observation.

Lemma 1. $R \lceil Q$, *the restriction of R to Q, is* ∃*-expansive in the poset* $(Q, \leq)$.

Proof of the lemma. Let $a \in Q$. As $R[a] \cap [a) \neq \emptyset$, there exists $b \in P$ such that

(1) $a\ R\ b$ and $a \le b$.

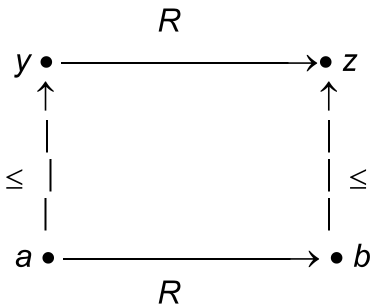
We prove that $b \in Q$. This will show that $R\lceil Q$ is $\exists$-expansive. By (1) and the monotonicity of R in $(P, \le)$ there exists $c \in P$ such that $b\ R\ c$ and $b \le c$ (see the figure below).



This means that $c \in R[b] \cap [b)$. Hence

(2) $R[b] \cap [b) \neq \varnothing$.

Now let y be an arbitrary element of P such that $R[y] \subseteq (y]$. We show $b \le y$. As $a \in Q$ and $R[y] \subseteq (y]$, we have that $a \le y$. As $a\ R\ b$, the monotonicity of R implies the existence of an element $z \in P$ such that $y\ R\ z$ and $b \le z$ (see the diagram below).



As $z \in R[y]$ and $R[y] \subseteq (y]$, we get that $z \le y$. Consequently, $b \le z \le y$ which gives that $b \le y$. This proves that $b \in Q$. □

If C is a well-ordered chain in $(P, \le)$ of type ω and $f: C \to P$ is a monotone mapping, then $f[C]$ is a chain of type $\le \omega$. Hence $\sup(f[C])$ exists in $(P, \le)$. Evidently, every chain in $(Q, \le)$ is also a chain in $(P, \le)$. Consequently, by the σ-continuity of R:

(3) *For every well-ordered chain C in $(Q, \le)$ of type ω and for every monotone mapping $f: C \to P$ such that $x\ R\ f(x)$ for all $x \in C$, there holds*: $\sup(C) R \sup(f[C])$.

Lemma 2. *Let C be a well-ordered chain in $(Q, \le)$ ot type ω. Assume that there* *exists a monotone mapping* $f : C \to Q$ *such that* $x \, R \, f(x)$ *and* $x \leq f(x)$ *for all* $x \in C$. *Then the supremum* sup (C) *belongs to* Q.

Proof of the lemma. Let $M := \sup(C)$. We show $M \in Q$. By (3) we have that $M \, R \, \sup(f[C])$. Furthermore, as $x \leq f(x)$ for all $x \in C$, it follows that $M = \sup(C) \leq \sup(f[C])$. This shows that $\sup(f[C]) \in R[M] \cap [M)$. Hence

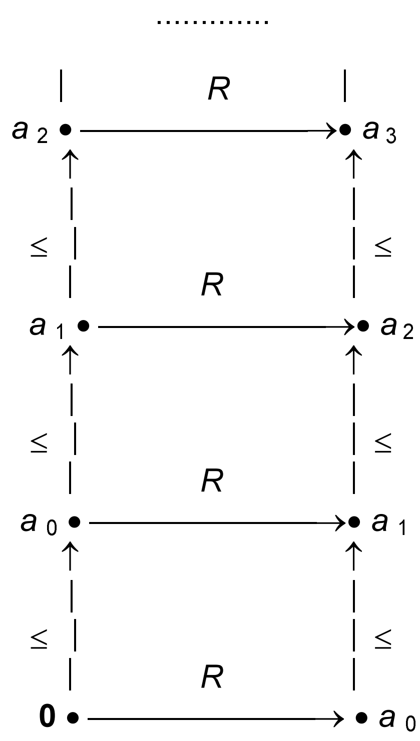
(4) $R[M] \cap [M) \neq \varnothing$.

Now let $y \in P$ be an element such that $R[y] \subseteq (y]$. We claim that $M \leq y$. As $C \subseteq Q$, we have that $x \leq y$ for all $x \in C$, by the second conjunct of the definition of Q. It follows that $M = \sup(C) \leq y$. This and (4) prove that $M \in Q$. □

We now pass to the proof of the theorem. We inductively define a strictly increasing sequence $a_0 < a_1 < \ldots < a_n < a_{n+1} < \ldots$ of elements of Q. The type of the sequence is $\leq \omega$.

We define: $a_0 := \mathbf{0}$. Suppose the elements $a_0 < a_1 < \ldots < a_n$ have been defined. As $a_n \in Q$ and, by Lemma 1, the relation $R \lceil Q$ is $\exists$-expansive, there exists an element $b \in Q$ such that $a_n \leq b$ and $a_n \, R \, b$. If $b = a_n$, the defining procedure terminates. In this case $a^* := a_n$ is already a fixed-point of R. As a^* belongs to Q, the second statement of the thesis of the theorem evidently holds for a^*. If $b \neq a_n$, we put: $a_{n+1} := b$. Clearly $a_n < a_{n+1}$.

It remains to consider the case when the sequence sequence $a_0 < a_1 < \ldots < a_n < a_{n+1} < \ldots$ has type ω. In this case we put $C := \{a_n : n \in \omega\}$ and define $f : C \to P$ by $f(a_n) := a_{n+1}$ for all $n \in \omega$. f is well-defined and monotone. As $C \subseteq Q$, $a \, R \, f(a)$ and $a \leq f(a)$ for all $a \in C$, the supremum $\sup(C)$ belongs to Q, by Lemma 2. Furthermore $\sup(C) \, R \, \sup(f[C])$, by the σ-continuity of R. But evidently $\sup(C) = \sup(f[C])$ because $a_0 = \mathbf{0}$ (see the figure below). Putting $a^* := \sup(C)$, we thus see that $a^* \, R \, a^*$. So a^* is a fixed-point of R. Since $a^* \in Q$, it follows that for every $y \in P$, if $R[y] \subseteq (y]$, then $a^* \leq y$. □



Notes. (1). The hypothesis of monotonicity in Theorem 4.2.7 is essential and cannot be dropped altogether. For let $P = [0,1]$ be the closed unit interval of real numbers. Evidently, the system $(P, \leq)$ with the usual ordering $\leq$ of real numbers is a chain-σ-complete poset. The relation $R \subseteq P \times P$ is defined as follows:

$$a\, R\, b \text{ iff } (\exists n \in \omega, n \geq 1)\ |a - b| = (1/2)^n.$$

It is easy to see that:

(a) R is symmetric and serial,
(b) R is not monotone,
(c) R does not posses a fixed-point.

As to (b), observe that for the numbers 1/2 and 1, we have $1/2\ R\ 1$ and $1/2 < 1$. But there does not exists a number d in [0,1] such that $1\ R\ d$ and $1 \leq d$.

(2). The hypothesis that $(P, \leq)$ is chain-σ-complete is essential in Theorem 4.2.7. For let $P = \{1, 2, a, b\}$, where $1 < 2$ and $a < b$. Let R be the relation on P defined as follows:

$$R := \{ \langle 1, a\rangle, \langle a, 1\rangle, \langle 2, b\rangle, \langle b, 2\rangle \}.$$

R is symmetric and monotone. R is also chain σ-continuous. But R does not possess a fixed-point. The poset $(P, \leq)$ is not chain-σ-complete because it does not have the zero element, the supremum of the empty chain. □

Let $(P, \leq)$ be a chain-σ-complete poset. A mapping $\pi: P \to P$ is called σ-*continuous* if it is monotone and

$$\pi(\sup(C)) = \sup(\pi[C])$$

for every non-empty and countable chain C in $(P, \leq)$.

Since for any monotone mapping $\pi: P \to P$, the image $\pi[C]$ of any chain C in $(P, \leq)$ is a chain as well, we see that, in view of the chain-σ-completeness of $(P, \leq)$, the above formula thus postulates the equality of the two suprema and not their existence.

It is clear that a mapping $\pi: P \to P$ is σ- continuous in the above sense iff the graph R_σ is σ-continuous as a binary relation.

Corollary 4.2.8. *Let* $(P, \leq)$ *be a chain-σ-complete poset. Every σ-continuous mapping* $\pi: P \to P$ *has a least fixed-point* a^*, *i.e.*, $\pi(a^*) = a^*$ *and*

$$(\forall\, b \in P)\ (\pi(b) \leq b \text{ implies } a^* \leq b).$$

Proof. We work with the graph R_π of π and proceed as the proof of Theorem 4.2.7. Since π is σ-continuous, the graph R_π is a chain- σ-continuous relation and $R_\pi[\mathbf{0}] = \{\pi(\mathbf{0})\}$. By Theorem 4.2.7, there is a fixed-point a^* of R_π such that for every $b \in P$, $R_\pi[b\,] \subseteq (b\,]$ implies that $a^* \leq b$. But the last condition simply says that for every $b \in P$, $\pi(b) \leq b$ implies $a^* \leq b$.

Let us also note that the chain $a_0 \leq a_1 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots$, defined as in the proof of Theorem 4.2.7 for the relation R_π, has the following properties:

$$a_0 = \mathbf{0},$$
$$a_{n+1} = \pi(a_n), \text{ for all } n.$$

Furthermore, $a^* = \sup(\{a_n : n \in \omega\})$. □

Some additional remarks are appropriate in the context of the above theorems. Apart from the notion of chain-σ-completeness of a poset, one can define the relative properties formulated in terms of countable well-ordered chains and countable directed subsets.

We say that a poset $(P, \leq)$ is:

(A) *well-ordered chain-σ-complete* if every countable chain C in $(P, \leq)$ of type ω has a supremum,

(B) *directed-σ-complete* if every countable directed subset D in $(P, \leq)$ has a supremum.

It is clear that the chain-σ-completeness implies property (A) and that (B) Implies the chain-σ-completeness. However, it readily follows from Theorem 4.1.1 that the above three properties are equivalent. In the sequel we will not carefully distinguish between these three situations. However, we shall uniformly formulate the results discussed in this chapter in terms of the chain-σ-completeness and inductivity of posets.

We shall discuss a certain property of relations which is stronger than monotonicity. It is called the transfer property for upper bounds of chains. Monotonicity is a special case of the transfer property obtained by restricting this property to one-element chains.

Definition 4.2.9. Let $(P, \leq)$ be a poset. We say that a binary relation $R \subseteq P \times P$ has *the transfer property for upper bounds of directed sets* if the following holds:

For every non-empty directed set D in $(P, \leq)$, for every monotone mapping $f : D \to P$ such that $x \; R \; f(x)$ for all $x \in D$, and for every upper bound a of D in $(P, \leq)$ there exists an element $b \in P$ such that

(i) b is an upper bound of the directed set $f[D]$ in $(P, \leq)$,

(ii) $a \; R \; b$. □

By confining the above property to certain types of directed subsets further refinements of the transfer property are obtained, as e.g. the *transfer property for upper bounds of chains* and the *transfer property for upper bounds of well-ordered chains*.

In the sequel we will be mainly concerned with chain-complete posets. Therefore the format of the transfer property will be mainly restricted to the contexts in which these posets occur. But the results presented below also holds for the other types of posets we have introduced.

Corollary 4.2.10. *If* $(P, \leq)$ *is a chain-complete poset. Let* $R \subseteq P \times P$ *be a binary relation. The following conditions ae equivalent*:

(1) *R has the transfer property for upper bounds of chains in* $(P, \leq)$.

(2) *R is monotone and R satisfies the following condition* :

(*) *For every non-empty chain C in* $(P, \leq)$, *for every monotone mapping* $f : C \to P$ *such that* $x \; R \; f(x)$ *for all* $x \in C$, *there exists an element* $b \in P$ *such that* $\sup(C) \; R \; b$ *and* $\sup(f[C]) \leq b$.

The analogous fact holds for directed-complete posets and well-ordered chain-complete posets, *respectively, with chains replaced by directed sets and well-ordered chains, respectively*. □

We note that if $\pi : P \to P$ is a monotone mapping (in the usual sense) then the graph R_π of π has automatically the transfer property. For let C be a non-empty chain in $(P, \leq)$. Assume $f : C \to P$ is a monotone mapping such that $x \; R_\pi \; f(x)$ for all $x \in C$ and let a be an upper bound of C. Evidently, $f(x) = \pi(x)$ for all $x \in C$. As $x \leq a$ for all $x \in C$, we obtain that $\pi(x) \leq \pi(a)$ for all $x \in C$, by the monotonicity of π. It follows that $b := \pi(a)$ is an upper bound of the chain $f[C]$ in $(P, \leq)$ and $a \; R_\pi \; b$.

We shall supplement the list of fixed-point theorems for relations with the following one. In this theorem, as compared with Theorem 4.2.7, a stronger property is assumed about the poset $(P, \leq)$, viz. chain-completeness. In turn, the property of chain-σ-continuity of a relation is replaced by the transfer property for upper bounds of chains.

Theorem 4.2.11. *Let* $(P, \leq)$ *be a chain-complete poset. Let* $R \subseteq P \times P$ *be a relation with the transfer property for upper bounds of chains such that the set* $R\,[\mathbf{0}]$ *is non-empty. Then R has a fixed-point a *. Furthermore, a * can be assumed to have the property: for every* $y \in P$*, if* $R\,[y\,] \subseteq (y\,]$*, then* $a^* \leq y$*.*

The analogous results hold for well-ordered chain-complete posets and directed-complete posets, respectively.

The proof much resembles the proof of Theorem 4.2.7. As R has the transfer property, R is a monotone relation in $(P, \leq)$.

We define:

$Q := \{x \in P : R\,[x\,] \cap [x\,) \neq \varnothing$ and $(\forall y \in P\,)\,(R\,[y\,] \subseteq (y\,]$ implies $x \leq y\,)\,\}$.

The set Q is non-empty because $\mathbf{0}$ is in it.

Lemma 1. $R \lceil Q$, *the restriction of R to Q, is* $\exists$*-expansive in the poset* $(Q, \leq)$.

This lemma is proved exactly in the same way as the corresponding Lemma 1 in the proof of Theorem 4.2.7. □

If C is a well-ordered chain in $(P, \leq)$ and $f : C \to P$ is a monotone mapping, then the chain $f\,[C\,]$ is also well-ordered. Consequently, $\sup\,(f\,[C\,])$ exists in $(P, \leq)$. Evidently, every chain in $(Q, \leq)$ is also a chain in $(P, \leq)$. Hence, by Corollary 4.2.10.(2):

(3) *For every well-ordered chain C in* $(Q, \leq)$ *and for every monotone mapping* $f : C \to P$ *such that* $x\,R\,f\,(x\,)$ *for all* $x \in C$*, there exists* $b \in P$ *such that* $\sup(f\,[C\,]) \leq b$ *and* $\sup(C)\;R\;b$.

Lemma 2. *Let C be a well-ordered chain in* $(Q, \leq)$*. Assume that there exists a monotone mapping* $f : C \to Q$ *such that* $x\,R\,f\,(x\,)$ *and* $x \leq f\,(x\,)$ *for all* $x \in C$*. Then* $\sup\,(C)$ *belongs to Q.*

Proof of the lemma. Let $M := \sup\,(C)$. By (3) there exists $b \in P$ such that $M\;R\;b$ and $\sup(f\,[C\,]) \leq b$. As $x \leq f\,(x\,)$ for all $x \in C$, it follows that $M = \sup\,(C) \leq \sup(f\,[C\,]\,) \leq b$. Hence $M \leq b$. This shows that

(5) $R\,[M\,] \cap [M\,) \neq \varnothing$.

Now let $y \in P$ be an element such that $R\,[y\,] \subseteq (y\,]$. We claim that $M \leq y$. As $C \subseteq Q$, we have that $x \leq y$ for all $x \in C$, by the second conjunct of the definition of Q. It follows that $M = \sup\,(C) \leq y$. This and (4) prove that $M \in Q$. □

We shall make use of the following easily provable fact.

Fact. *For every poset* $(Q, \leq)$ *there exists a least ordinal* μ *such that the type of each well-ordered chain in* $(Q, \leq)$ *is equal or smaller than* μ. □

The following lemma is crucial:

Lemma 3. *There exists a well-ordered chain*

$$\{a_\alpha : \alpha \in Ord\}$$

of elements of Q such that

(5) $\quad a_\alpha R a_{\alpha+1}$

for all ordinals α.

The proof is similar to the proof of Theorem 4.2.7. The sequence a_α, $\alpha \in Ord$, is defined by transfinite induction.

For $\alpha = 0$ we put

$$a_0 := \mathbf{0} \text{ (= the least element of Q).}$$

Let β be an ordinal, $\beta > 0$. Asume that the chain

$$\{a_\alpha : \alpha \leq \beta\}$$

has been defined so that (5) holds for all $\alpha < \beta$. As a_β belongs to Q, Lemma 1 implies that there exists $b \in Q$ such that $a_\beta R b$ and $a_\beta \leq b$. We then put

$$a_{\beta+1} := b.$$

Now let λ be a limit ordinal. Assume (5) holds for the elements of the well-ordered chain

$$C := \{a_\alpha : \alpha < \lambda\} \subseteq Q,$$

i.e., $a_\alpha R a_{\alpha+1}$ for all ordinals $\alpha < \lambda$. Let $M := \sup(C)$. We show M belongs to Q.

Define the mapping $f : C \to Q$ as follows. Let $x \in C$. So $x = a_\alpha$ for the smallest possible $\alpha < \lambda$. Then define $f(x) := a_{\alpha+1}$. The mapping f is well-defined. Furthermore $x R f(x)$ and $x \leq f(x)$ for all $x \in C$. We claim that f is monotone. Indeed, assume $x, y \in C$ and $x < y$. We have that $x = a_\alpha$ and $y = a_\beta$ for smallest possible α, $\beta < \lambda$. As $a_\alpha < a_\beta$ we have that $\alpha \leq \beta$. (For otherwise suppose that $\alpha > \beta$. Then the definition of C gives that $a_\beta \leq a_\alpha$, i.e., $y \leq x$, a contradiction.) As $\alpha \leq \beta$, we get $\alpha + 1 \leq \beta + 1$. Hence $a_{\alpha+1} \leq a_{\beta+1}$, i.e., $f(x) \leq f(y)$.

Applying Lemma 2 to the above situation, we infer that $M \in Q$. We then put :

$$a_\lambda := M.$$

This concludes the proof of Lemma 3. □

Since Q is a set, the above fact implies that there exists an ordinal μ which stabilizes the sequence a_α, $\alpha \in Ord$. This means that $a_\alpha = a_{\alpha+1}$ for all $\alpha \geq \mu$. Putting $a^* := a_\mu$, we obtain that $a^* = a_\mu \, R \, a_{\mu+1} = a_\mu = a^*$, i.e., $a^* \, R \, a^*$. So a^* is a fixed-point of R.

The second thesis of Theorem 4.2.11 directly follows from the definition of Q. □

Corollary 4.2.12. *Let $(P, \leq)$ be an inductive poset. Let $\pi : P \to P$ be a monotone mapping. Then π has the least fixed-point, i.e., there exists a^* in P such that $\pi(a^*) = a^*$ and for every $y \in P$, if $\pi(y) \leq y$, then $a^* \leq y$.*

An analogous result holds for directed-complete posets and well-ordered chain-complete posets, respectively.

Proof. As π is monotone, the graph of π satisfies the hypotheses of Theorem 4.2.11. □

Theorem 4.2.11 and Corollary 4.2.12 bear interesting consequences for the theory of infinitistic definitions. In the proof of Theorem 4.2.11 (see Lemma 3) the technique of definability by means of transfinite recursion was applied. This method is formalized in the form of the general Principle of Definability by Transfinite Recursion. We state this principle below. (It is not difficult to show that the proof of Lemma 3 can be strictly formalized in terms of this principle.) But, more interestingly, the converse of this situation holds: the Principle of Definability by Transfinite Recursion can be deduced from Corollary 4.2.12. This is the content of the next theorem.

Theorem 4.2.13. (The Principle of Definability by Transfinite Recursion). *Let α be an ordinal number and let X be a set. Suppose G is a function defined on the set $\cup_{\beta < \alpha} X^\beta$ with values in X. Then there exists a unique function $F : \alpha \to X$* such that

(1) $\quad F(\beta) = G(F \lceil \beta)$ *for all $\beta < \alpha$.*

Proof. We put:

$$P_0 := \cup_{\beta < \alpha} X^\beta .$$

Thus $G : P_0 \to X$. Furthermore, we define:

$$P := \cup_{\beta \leq \alpha} X^\beta .$$

The set P ordered by inclusion forms a directed-complete poset. (Each function $f \in X^\beta$ is a subset of $\beta \times X$.) **0**, the empty function, is the smallest element in $(P, \subseteq)$. **0** is the sole element of X^0.

For each function $f \in P$ we define the function $\pi(f) \in P$ as follows. We consider two cases.

Case 1. $f \in P_0$.

The domain of f, $Dom(f)$, is an ordinal, say β, strictly smaller than α. We then put:

(2) $Dom(\pi(f)) := \beta + 1$ and $\pi(f)(\gamma) := G(f\lceil\gamma)$ for all $\gamma \leq \beta$.

In Case 1 we have that $f\lceil\gamma \in P_0$ for all $\gamma \leq \beta$. Hence G is defined at $f\lceil\gamma$ for all $\gamma \leq \beta$. This shows that $\pi(f)$ is well-defined. We also note that $\pi(f)(\beta) = G(f)$ and $\pi(f)(0) = G(\mathbf{0})$.

Case 2. $f \in P - P_0$.

Then $Dom(f) = \alpha$ and we put:

(3) $Dom(\pi(f)) := \alpha$ and $\pi(f)(\gamma) := G(f\lceil\gamma)$ for all $\gamma < \alpha$.

In this case, as $f : \alpha \to X$, we have that $f\lceil\gamma \in P_0$ for all $\gamma < \alpha$. Hence $\pi(f)$ is well-defined.

Claim 1. *The mapping $\pi : P \to P$ is monotone.*

Proof of the claim. We observe that π acts as a substitution. For each $f \in P$ and each γ belonging to the domain of f, π replaces the value $f(\gamma)$ by $G(f\lceil\gamma)$. If, furthermore, $f \in P_0$ and hence $Dom(f) = \beta$ is smaller than α, π adjoins the pair $\{\langle\beta, G(f)\rangle\}$ to the graph $\{\langle\gamma, G(f\lceil\gamma)\rangle : \gamma < \beta\}$. It is then clear if a larger $g \supseteq f$, we get a larger function $\pi(g)$. □

Claim 2. *If f is a fixed-point of π, then f is a total function*, *i.e.*, $f : \alpha \to X$.

Proof of the claim. Suppose $Dom(f)$ is an ordinal β smaller than α. But, as $\beta < \alpha$, G is defined at $f\lceil\beta$. Hence $\pi(f)$ is defined at β (and $\pi(f)(\beta) = G(f)$). As $\pi(f) = f$, it follows that f itself is defined at β. But this is impossible because $\beta \notin \beta = Dom(f)$. □

In view of Corollary 4.2.12, π has the least fixed-point, say F. But it follows from Claim 2 that F is the sole fixed-point of π. Indeed, suppose g is a fixed-point of π. As F is the least fixed-point, we have that $F \subseteq g$. But, by Claim 2, $F : \alpha \to X$. Hence $F = g$. This proves that F is a unique function $F : \alpha \to X$ which satisfies (1). □

We prove that the logical status of Theorem 4.2.11, Corollary 4.2.12 and of the Principle of Definability of Transfinite Recursion (PDTR) is the same – on the basis of modest set-theoretic assumptions all the above statements are deductively equivalent. But one thing should be clarified here – PDTR is provable in the Zermelo-Fraenkel set theory. It follows by the laws of classical logic that if φ and ψ are theorems of any theory, say T, then so is the sentence $\varphi \leftrightarrow \psi$. Hence, as Theorem 4.2.11, Corollary 4.2.12 and PDRT are provable in ZF, it trivially follows that they are deductively equivalent within ZF. But if one takes a weaker theory T than ZF, PDRT need not be provable on the basis of T. For example, the „standard" proof of PDRT makes a heavy use of the Principle of Transfinite Induction. The latter principle (restricted to elementary properties of sets) is provable in ZF. But the proof of the Principle of Transfinite Induction itself rests on the Axiom of Subsets.

Theorem 4.2.14. *On the basis of some weak set-theoretic assumptions*, *Theorem* 4.2 11, *Corollary* 4.2.12 *and The Principle of Definability by Transfinite Recursion are deductively equivalent.*

Proof. To simplify the matters, *T* is assumed to be a fragment of ZF consisting of the Axiom of the Empty Set, the Axiom of Pairs, The Axiom of Union, the Powerset Axiom and the Axiom of Extensionality. Furthermore *T* involves a restricted form of the Replacement Axiom (and of the Axiom of Subsets in particular) – rich enough so as to prove the existence of subsets needed in the proofs of the below implications. In fact, only finitely many instances of the scheme provided by the Replacement Axiom is needed here. One of such instances guarantees, in the presence of the remaining axioms, the existence of the Cartesian product of two sets *X* and *Y* (see e.g. Kuratowski and Mostowski [1967]) In the standard language of set theory $L = \{\in\}$ we define predicates such as *Ord*(*x*), read as: "*x* is an ordinal", *Funct* (*x*), read as: "*x* is a funcion", the ternary relation *R*(*x*, *y*, *f*), read as: „"*f* is a function from *x* to *y*". (The above axioms do not preclude the existence of some sets. But the language *L* is so capacious that we may speak of convenient descriptions of some set theoretic entities leaving aside, in some situations, the question of the existence of sets satisfying these descriptions.)

Assume *T*. Evidently, $T \vdash$ Theorem 4.2.11 $\rightarrow$ Corollary 4.2.12. The proof of Theorem 4.2.11 shows that $T \vdash$ PDTR $\rightarrow$ Theorem 4.2.11. In turn, the proof of Theorem 4.2.13 shows that $T \vdash$ Corollary 4.2.12 $\rightarrow$ PDTR. □

We thus see that in the group of fixed point-theorems we haxe examined at the end of this section, each of them is comparable in strength to the Principle of Definablity by Transfinite Induction. A similar remark can be said about Theorem 4.2.7 and Corollary 4.2.8: each of this results is equivalent to the purely arithmetic Princible of Definability by Recursion.

Summing up the results of this paragraph, we may say that the fixed-point theorems we have discussed *do not provide essentially new logical tools* for definablity of objects with various desired properties. Modulo some weak set-theoretic or arthmetic assumptions, each of these fixed-point theorems is either equivalent to the Axiom of Choice or to the well-knownPrinciple of Definability by Transfinite Recursion or it is equivalent to the usual arithmetic Principle of Definability by Recursion. The significance of fixed-point theorems lies rather in their elegance, brevity and adaptability to various situations in which one rather directly operates with ordered structures than sets (or classes) of natural or ordinal numbers.

4.3. The Caristi-Kirka Fixed-Point Theorem.

Let (*X*, *d*) be metric space with a metric *d*. Thus *d* is a function which assigns to any pair *x*, *y* of elements of *X* a non-negative real number *d* (*x*, *y*), their *distance* (or *separation*), such that the following conditions hold, for any $x, y, z \in X$:

(1) $d(x, y) = 0$ if and only if $x = y$ (reflexivity),

(2) $d(x, y) = d(y, x)$ (symmetry),

(3) $d(x, z) \leq d(x, y) + d(y, z)$ (the triangle inequality).

R stands for the set of real numbers. A real-valued function $t : X \to \mathbf{R}$ is said to be *lower-continuous* if, for every $r \in \mathbf{R}$, the set $\{x \in X : t(x) \leq r\}$ is closed in the space (X, d).

A function $t : X \to \mathbf{R}$ is *lower-bounded* if the set $\{ t(x) : x \in X \}$ has a lower bound in ***R*** (or, equivalently, $\inf\{ t(x) : x \in X \}$ exists in ***R***).

Let $t : X \to \mathbf{R}$ be a real-valued function. We define the following binary relation $\leq_t$ on the space X::

$$x \leq_t y \text{ if and only if } d(x, y) \leq t(x) - t(y),$$

for all $x, y \in X$.

Theorem 4.3.1. *Let* (X, d) *be metric space with a metric d. Let* $t : X \to \mathbf{R}$ *be a function. Then the following conditions hold* :

(A) $\leq_t$ *is an order on* X .

(B) *Let C be a non-empty subset of X. The set C is a chain in the poset* $(X, \leq_t)$ *iff for every pair* $x, y \in C$, $d(x, y) \leq |t(x) - t(y)|$.

($|r|$ *is the absolute value of the number r* .)

(C) *If the metric space* (X, d) *is complete and* $t : X \to \mathbf{R}$ *is lower-continuous and lower-bounded, then in the poset* $(X, \leq_t)$ *every non-empty chain has an upper bound*.

(D) *If, furthermore,* $t : X \to \mathbf{R}$ *is continuous, then in the poset* $(X, \leq_t)$ *every non-empty chain has a supremum.*

Proof. We first note that the above definitions yield that if $x \leq_t y$, then $t(y) \geq t(x)$. Furthermore, if $t(x) = t(y)$ and $x \neq y$, then x and y are incomparable with respect to the order $\leq_t$.

We also observe that in (3) it may be assumed without loss of generality that t is non-negative, that is, t maps X into the set $\mathbf{R}^+ := \{r \in \mathbf{R} : r \geq 0\}$ of non-negative reals. Indeed, if $t : X \to \mathbf{R}$ is lower-bounded and $m := \inf \{ t(x) : x \in X \}$, then the function t^+ defined by

$$t^+(x) := t(x) + m, \quad x \in X,$$

maps X into the set $\mathbf{R}^+$, $t^+ : X \to \mathbf{R}^+$. It is clear that t is lower-continuous (continuous) iff t^+ has the same property. But, more importantly, the order relation $\leq_t$

determined by t coincides with the order determined by t^+, as can be easily checked.

(A). $\leq_t$ *is reflexive*.

Let $a \in X$. We must show that $d(a, a) \leq t(a) - t(a)$. But this holds by the reflexivity of d.

$\leq_t$ *is transitive*.

Assume that $a \leq_t b$ and $b \leq_t c$ for some $a, b, c \in X$. This means that $d(a, b) \leq t(a) - t(b)$ and $d(b, c) \leq t(b) - t(c)$. Adding these inequalities, we get

$$d(a, b) + d(b, c) \leq (t(a) - t(b)) + (t(b) - t(c)) = t(a) - t(c).$$

Since $d(a, c) \leq d(a, b) + d(b, c)$, we thus have that $d(a, c) \leq t(a) - t(c)$. So $a \leq_t c$.

$\leq_t$ *is antisymmetric*.

Assume that $a \leq_t b$ and $b \leq_t a$ for some $a, b \in X$. This means that $d(a, b) \leq t(a) - t(b)$ and $d(b, a) \leq t(b) - t(a)$. Adding these inequalities and using the symmetry of d, we obtain that $0 \leq 2d(a, b) \leq (t(a) - t(b)) + (t(b) - t(a)) = 0$. Hence $d(a, b) = 0$ and, consequently, $a = b$.

So (A) holds.

(B). ($\Rightarrow$). Assume C is a chain in $(X, \leq_t)$. Hence, for any $a, b \in C$,

(*) either $d(a, b) \leq t(a) - t(b)$ or $d(a, b) \leq -(t(a) - t(b))$.

As $0 \leq d(a, b)$, in both cases we have that $d(a, b) \leq |t(a) - t(b)|$.

($\Leftarrow$). To prove the reverse implication, assume that $d(a, b) \leq |t(a) - t(b)|$, for all $a, b \in C$. If $|t(a) - t(b)| = t(a) - t(b)$, we have that $d(a, b) \leq t(a) - t(b)$, which means that $a \leq_t b$. In the other case, if $|t(a) - t(b)| = -(t(a) - t(b))$, by symmetry of d we get that $d(b, a) \leq t(b) - t(a)$. So $b \leq_t a$.

(C). Let C be a non-empty chain in $(X, \leq_t)$. As t is lower-bounded, there exists a real number $m := \inf\{ t(x) : x \in C\})$. Hence, for every natural n, $n \geq 1$, there exists an element $x_n \in C$ such that $m \leq t(x_n) < m + 1/n$. (The Countable Principle of Choice AC_N is used here.)

Claim 1. $\{x_n\}$ *is a Cauchy sequence in the metric space* (X, d).

Proof of the claim. For any natural k, l we have:

$$|t(x_k) - t(x_l)| = |(t(x_k) - m) - (t(x_l) - m)| \leq |t(x_k) - m| + |t(x_l) - m| \leq 1/k + 1/l$$

On the other hand, the set $\{x_n : n \geq 1\}$, being a subset of C, is also a chain. Hence, by (B),

$$d(x_k, x_l) \leq |t(x_k) - t(x_l)| \text{ for all } k, l.$$

It follows that $d(x_k, x_l) \leq 1/k + 1/l$ for all k, l. This proves the claim.

Since the space (X, d) is complete, Claim 1 implies that the sequence $\{x_n\}$ is convergent in the metric d to some point $x^* \in X$.

Claim 2. $t(x^*) \leq m$.

Proof of the claim. For every $\varepsilon > 0$ define

$$G_\varepsilon := \{x \in X : t(x) > t(x^*) - \varepsilon\}.$$

Evidently, $x^* \in G_\varepsilon$. Since t is lower-continuous, G_ε is open in (X, d). Hence, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\{x \in X : d(x, x^*) < \delta\} \subseteq G_\varepsilon$. As x^* is the limit of the sequence $\{x_n\}$ in the metric d, it follows that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for almost all n the element x_n belongs to $\{x \in X : d(x, x^*) < \delta\}$ and hence it belongs to G_ε. Thus, for every $\varepsilon > 0$, almost all elements of $\{x_n\}$ belong to G_ε. This means that for every $\varepsilon > 0$ and for almost all n,

$$t(x^*) < t(x_n) + \varepsilon.$$

But it follows from the definition of the sequence $\{x_n\}$ that m is the limit of the number sequence $\{t(x_n)\}$, i.e., $m = \lim_{n \to \infty} t(x_n)$. This fact and the above inequality yield that $t(x^*) \leq \lim_{n \to \infty} t(x_n) = m$. This proves the claim.

Claim 3. *The element x^* is an upper bound of the chain C.*

Proof of the claim. Suppose x^* is *not* an upper bound of C. Hence there exists $x_0 \in C$ such that it is not the case that $x_0 \leq_t x^*$. This, in virtue of the definition of $\leq_t$, gives that $d(x_0, x^*) > t(x_0) - t(x^*)$. Hence

$$d(x_0, x^*) + t(x^*) = t(x_0) + \delta$$

for some $\delta > 0$. Fix this δ. Applying Claim 2 we thus obtain that

$$d(x_0, x^*) + m \geq d(x_0, x^*) + t(x^*) = t(x_0) + \delta$$

and hence

(1) $\qquad d(x_0, x^*) + m \geq t(x_0) + \delta.$

But as n grows large, the number $d(x_0, x_n)$ is close to $d(x_0, x^*)$, by the continuity of d. It follows that for almost all n,

$(2)_n$ $\qquad d(x_0, x_n) \geq d(x_0, x^*) - \delta/2,$

i.e., there exists n_0 such that $(2)_n$ holds for all $n \geq n_0$. Taking then into account the fact that $t(x_n) \geq m$ for all n, we deduce from $(2)_n$ that

$$(3)_n \qquad d(x_0, x_n) + t(x_n) \geq d(x_0, x^*) + m - \delta/2$$

for almost all n. $(3)_n$ and (1) yield that

$$(4)_n \qquad d(x_0, x_n) + t(x_n) \geq t(x_0) + \delta/2$$

for almost all n. Whence

$$d(x_0, x_n) + t(x_n) > t(x_0)$$

for almost all n, which means that it is not the case that $x_0 \leq_t x_n$ for almost all n. This and the fact that C is a chain implies that $x_n <_t x_0$ for almost all n, i.e., $d(x_0, x_n) < t(x_n) - t(x_0)$. Applying then $(4)_n$ to this situation, we obtain that

$$t(x_n) - t(x_0) > d(x_0, x_n) \geq t(x_0) - t(x_n) + \delta/2$$

and hence

$$t(x_n) - t(x_0) > t(x_0) - t(x_n) + \delta/2$$

for almost all n. Simplifying the last inequality, we get that

$$t(x_n) > t(x_0) + \delta/4$$

for almost all n. Passing to the limit, we obtain $m = \lim_{n\to\infty} t(x_n) \geq t(x_0) + \delta/4$. Hence $m > t(x_0)$. As $x_0 \in C$, the obtained inequality contradicts the fact that $m = \inf\{t(x) : x \in C\}$). The arrived contradiction proves the claim.

The above chain of claims concludes the proof of (C).

(D). We additionally assume that t is continuous.

Claim 4. $m = t(x^*)$.

Proof of the claim. We have $m = \lim_{n\to\infty} t(x_n)$. Since t is continuous and $\lim_{n\to\infty} x_n = x^*$ in the metric d, we have that $m = \lim_{n\to\infty} t(x_n) = t(x^*)$. This proves the claim.

Claim 5. $x^* = \sup(C)$.

Proof of the claim. In view of Claim 3 it suffices to show that $x^* \leq_t y$ for every upper bound y of C. So assume that $d(x, y) \leq t(x) - t(y)$, for all $x \in C$. In particular, $d(x_n, y) \leq t(x_n) - t(y)$, for all $n \geq 1$ for the sequence $\{x_n\}$ defined as above. Passing to the limit we obtain: $d(x^*, y) = \lim_{n\to\infty} d(x_n, y) \leq \lim_{n\to\infty}(t(x_n) - t(y)) = t(x^*) - t(y)$ which means that $x^* \leq_t y$. This proves the claim and at the same time concludes the proof of (D).

The proof of the theorem is completed. □

It should be noted that condition (D) of Theorem 4.3.1 does not imply that the poset $(X, \leq_I)$ is inductive because it need not posses the least element. Hence the supremum of the empty chain need not exist in $(X, \leq_I)$. For example, let X be the set $\boldsymbol{R}$ of real numbers. X is endowed with the usual Euclidean metric: $d(x, y) := |x - y|$, for all $x, y \in X$. Let $t(x) := |x|$, for all $x \in X$. The function t is evidently continuous on X and lower-bounded (by 0). It immediately follows from the definition of $\leq_I$ and the properties of the absolute value function that the order $\leq_I$ on X determined by t is characterized by the equivalence: for any $x, y \in X$,

$$x \leq_I y \text{ iff } |x - y| = |x| - |y|.$$

Evidently, the number 0 is the greatest element in X the sense of $\leq_I$. Furthermore, for any $x, y \in X$, $x \leq_I y$ iff $-x \leq_I -y$. According to condition (D), every non-empty chain in $(X, \leq_I)$ has a supremum. The poset $(X, \leq_I)$ is not inductive, however, because it lacks the smallest element. □

We now pass to the proof of the main result of this section:

Theorem 4.3.2. (Caristi-Kirka Fixed-Point Theorem). *Let (X, d) be a complete metric space. Let $t : X \to \boldsymbol{R}$ be a lower-continuous and lower-bounded function. Then every function $\pi : X \to X$ which satisfies the following condition* :

(*) $\quad d(x, \pi(x)) \leq t(x) - t(\pi(x))$ *for all* $x \in X$,

has a fixed-point .

The burden of the proof of rests on the previous theorem. Define the poset $(X, \leq_I)$ as above. Theorem 4.3.1 guarantees that every non-empty chain in $(X, \leq_I)$ has an upper bound. Define the relation $R \subseteq X \times X$ as follows:

$$a\, R\, b \text{ iff } a \leq_I b \text{ and } a \leq_I \pi(b),$$

for all $a, b \in X$. (*) states that $a \leq_I \pi(a)$, for all $a \in X$. It follows that the relation R is $\forall$-expansive in the poset $(X, \leq_I)$. Furthermore, *every* element of X is a fixed-point of R. But Theorem 4.2.1 implies that R has a fixed-point a^* with the additional property that for every $b \in X$, $a^* R\, b$ implies $b = a^*$. As $a^* R\, \pi(a)$ by (*), it follows that $a^* = \pi(a^*)$. □

Remarks.

(1). Theorem 4.3.2 holds under the following weaker assumption made about the function π:

(**) $\quad (\forall x \in X)(\exists y \in X)\; d(x, y) \leq t(x) - t(y) \wedge d(x, \pi(y)) \leq t(x) - t(\pi(y))$.

Indeed, define the relation $R \subseteq X \times X$ by:

$$a\,R\,b \text{ iff } a \le_t \pi(b),$$

for all $a, b \in X$. (**) states that the relation R is $\exists$-expansive in the poset $(X, \le_t)$. By Theorem 4.2.2, R has a fixed-point a^* with the property:

(1) $(\forall b \in X)\; a^* R\, b \wedge a^* \le_t b \rightarrow b = a^*$.

The fact that a^* is a fixed-point of R means that

(2) $a^* \le_t \pi(a^*)$.

As R is $\exists$-expansive, there exists $b \in X$ such that $\pi(a^*)\, R\, b$ and $\pi(a^*) \le_t b$, which means that

(3) $\pi(a^*) \le_t b$ and $\pi(a^*) \le_t \pi(b)$.

But (3) and (2) imply that $a^* \le_t b$ and $a^* \le_t \pi(b)$, i.e., $a^* \le_t b$ and $a^* R\, b$. Applying (1) to this situation we obtain that $b = a^*$. This and the first conjunct of (3) give that $\pi(a^*) \le_t a^*$. Combining this with (2), we obtain that $a^* = \pi(a^*)$. So a^* is a fixed-point of π.

(2). Also dual versions of Theorems 4.3.1- 4.3.2 and of the definitions that precede them are interesting in their own right. Given a metric space (X, d) and a mapping $t : X \rightarrow \boldsymbol{R}$, we define the following binary relation $\le^d_t$ on the space X::

$$x \le^d_t y \text{ if and only if } d(x, y) \le t(y) - t(x),$$

for all $x, y \in X$. Consequently, $x \le^d_t y$ iff $y \le_t x$, for all $x, y \in X$.

The relation $\le^d_t$ coincides with the order relation $\le_{-t}$ determined by the function $-t$, i.e., $\le^d_t = \le_{-t}$, where $(-t)(x) := -t(x)$, for all $x \in X$.

A real-valued function $t : X \rightarrow \boldsymbol{R}$ is said to be *upper-continuous* if, for every $r \in \boldsymbol{R}$, the set $\{x \in X : t(x) \ge r\}$ is closed in the space (X,d).

A function $t : X \rightarrow \boldsymbol{R}$ is *upper-bounded* if the set $\{t(x) : x \in X\}$ has an upper bound in $\boldsymbol{R}$ (or, equivalently, $\sup\{t(x) : x \in X\}$ exists in $\boldsymbol{R}$).

Theorem 4.3.1*. *Let (X, d) be metric space with a metric d. Let $t : X \rightarrow \boldsymbol{R}$ be a function. Then the following conditions hold* :

(A)* $\le^d_t$ *is an order on* X.

(B)* *Let C be a non-empty subset of X. The set C is a chain in the poset $(X, \le^d_t)$ iff for every pair $x, y \in C$, $d(x, y) \le |t(x) - t(y)|$. Consequently, the orders $\le^d_t$ and $\le_t$ have the same chains (but $\le^d_t$ reverses the order in any chain in the sense of $\le_t$).*

(C)* *If the metric space (X, d) is complete and $t : X \rightarrow \boldsymbol{R}$ is upper-continuous and upper-bounded, then in the poset $(X, \le^d_t)$ every non-empty chain has an upper bound.*

(D)* *If, furthermore,* $t : X \to \mathbf{R}$ *is continuous, then in the poset* $(X, \leq^{d}_{t})$ *every non-empty chain has a supremum.*

The proof of Theorem 4.3.1* is similar to the proof of its prototype. In the proof of (C)* one argues as follows. Let C be a non-empty chain in $(X, \leq^{d}_{t})$. Since t is upper-bounded, there exists a real number $M := \sup\{t(x) : x \in C\}$). Hence, for every natural n, $n \geq 1$, there exists an element $x_n \in C$ such that $M - 1/n < t(x_n) \leq M$. Then consider the sequence $\{x_n\}$. □

The dual version of Theorem 4.3.2 takes the form:

Theorem 4.3.2*. *Let* (X, d) *be a complete metric space. Let* $t : X \to \mathbf{R}$ *be a upper-continuous and upper-bounded function. Then every function* $\pi : X \to X$ *which satisfies the following condition* :

(*) $d(x, \pi(x)) \leq t(\pi(x)) - t(x)$, *for all* $x \in X$,

has a fixed-point. □

Let (X, d) be a metric space. A mapping $\pi : X \to X$ is called *contractive* (or π is simply a *contraction*) if there exists a positive real number k, $0 < k < 1$, such that

$$d(\pi(x), \pi(y)) \leq k \cdot d(x, y),$$

for all $x, y \in X$.

It is evident that every contraction is continuous (in the metric d). The following observation is crucial:

Proposition 4.3.3. *Let* $\pi : X \to X$ *be a contraction of* (X, d). *Define the mapping* $t : X \to \mathbf{R}$ *as follows* :

$$t(x) := d(x, \pi(x)) / 1 - k .$$

for every $x \in X$. *The function* t *is continuous. Furthermore*

(1) $\inf\{t(x) : x \in X\} \geq 0$,

(2) $d(x, \pi(x)) \leq t(x) - t(\pi(x))$ *for all* $x \in X$.

Proof. Since both π and d are continuous, the continuity of t follows. (1) trivially holds because each metric is non-negative. To prove (2), we notice that the fact that π is a contraction implies that

$$d(\pi(x), \pi(\pi(x))) \leq k \cdot d(x, \pi(x)),$$

for all $x \in X$. Let $x \in X$. We then compute:

$$t(x) - t(\pi(x)) = [d(x, \pi(x)) / 1 - k] - [d(\pi(x), \pi(\pi(x))) / 1 - k] =$$

$[d(x, \pi(x)) - d(\pi(x), \pi(\pi(x)))] / 1 - k \geq [d(x, \pi(x)) - k \cdot d(x, \pi(x))] / 1 - k =$

$d(x, \pi(x))$.

Thus $t(x) - t(\pi(x)) \geq d(x, \pi(x))$ for every $x \in X$. □

We arrive at the following famous theorem of the fixed-points theory :

Theorem 4.3.4. (Banach Fixed Point Theorem). *Let* (X, d) *be a complete metric space. Then every contraction* $\pi : X \to X$ *has a unique fixed-point*.

Proof. $t : X \to \boldsymbol{R}$ be defined as in Proposition 4.3.3. The theorem follows from the thesis of Proposition 4.3.3 and Theorem 4.3.1. The uniqueness of a fixed-point trivially follows from the fact that π is a contraction. □

The above theorem proves useful in many areas of mathematics. Here we present a cluster of remarks concerning applicability of this theorem to a solution of the Cauchy problem in the theory of differential equations. These remarks play merely an illustrative role in the context of this book.

The n-th power $\boldsymbol{R}^n$ of the set $\boldsymbol{R}$ of real numbers, where $n \geq 1$, is defined inductively as follows. $\boldsymbol{R}^1 := \boldsymbol{R}$ and $\boldsymbol{R}^{n+1} := \boldsymbol{R}^n \times \boldsymbol{R}$. The set $\boldsymbol{R}^n$ is identified with the set of all n-tuples of real numbers. For every n - tuple $(x_1, \ldots, x_n) \in \boldsymbol{R}^n$ we define:

$$||x|| := |x_1| + \ldots + |x_n|.$$

Furthermore, for any two n – tuples $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of real numbers we put :

$$x - y := (x_1 - y_1, \ldots, x_n - y_n).$$

Let $n \geq 1$ be a fixed natural number. Fix a closed interval $[a, b]$ of real numbers and an element $(t_0, x_0) \in [a, b] \times \boldsymbol{R}^n$, where $x_0 = (x_{0,1}, \ldots, x_{0,n})$. Furthermore, let $f := (f_1, \ldots, f_n)$ be an n-tuple of mappings, where $f_i : [a, b] \times \boldsymbol{R} \to \boldsymbol{R}$ for $i = 1, \ldots, n$. f may be therefore identified with the mapping $f : [a, b] \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ assigning to each $(n + 1)$ - tuple $(t, x_1, \ldots, x_n) \in [a, b] \times \boldsymbol{R}^n$ the n - tuple $(f_1(t, x_1), \ldots, f_n(t, x_n))$. The Cauchy problem is formulated here in the form:

(*) $\quad z'(t) = f(t, z(t)), \quad z(t_0) = x_0.$

In other words, we ask about the existence of n differentiable functions $z_i : [a, b] \to \boldsymbol{R}$ such that $z_i'(t) = f_i(t, z_i(t))$ and $z_i(t_0) = x_{0,i}$ for $i = 1, \ldots, n$ and all $t \in [a, b]$.

The solution of the problem is provided by the following theorem due to Picard and Lindelöf:

Theorem 4.3.5. *Suppose that the following conditions hold* :

(1) $\quad f : [a, b] \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ *is a continuous mapping*.

(2) *There exists a positive real number L such that*

$$\|f(t, x) - f(t, y)\| \le L\|x - y\| \text{ on the set } [a, b] \times \boldsymbol{R}^n.$$

(3) $(t_0, x_0) \in [a, b] \times \boldsymbol{R}^n$.

Then the Cauchy problem (*) *has a unique solution*.

We shall outline the proof of this theorem. It is easy to see that the problem (*) is equivalent (in the obvious meaning of the word) to the following integral equation

$$(**) \qquad u(t) = f(t, x_0 + \int_{t_0}^{t} u(s)\, ds) \text{ for all } t \in [a, b].$$

We let $C([a, b], \boldsymbol{R}^n)$ denote the linear space of all continuous mappings $z : [a, b] \times \boldsymbol{R}^n \to \boldsymbol{R}^n$. Fix a positive real number λ. The space $C([a, b], \boldsymbol{R}^n)$ is endowed with the norm $\|\bullet\|_\lambda$, where

$$\|z\|_\lambda := \max\{\|z(t)\| \cdot \exp(-\lambda|t - t_0|) : t \in [a, b]\}.$$

The key point is that $C([a, b], \boldsymbol{R}^n)$ with the norm $\|\bullet\|_\lambda$ is a Banach space.

We then define the mapping $F : C([a, b], \boldsymbol{R}^n) \to C([a, b], \boldsymbol{R}^n)$ according to the formula: for every $u \in C([a, b], \boldsymbol{R}^n)$

$$(4) \qquad (Fu)(t) := f(t, x_0 + \int_{t_0}^{t} u(s)\, ds) \text{ for all } t \in [a, b].$$

It is not difficult to verify that Fu indeed belongs to $C([a, b], \boldsymbol{R}^n)$, for all $u \in C([a, b], \boldsymbol{R}^n)$. A straightforward computation also shows that for any $u, v \in C([a, b], \boldsymbol{R}^n)$,

$$(5) \qquad \|Fu - Fv\|_\lambda \le (L / \lambda) \cdot \|u - v\|_\lambda .$$

Selecting λ so that $\lambda > L$ and putting $k := L / \lambda$, we see that F is a contractive mapping of the Banach space $C([a, b], \boldsymbol{R}^n)$ (with the norm $\|\bullet\|_\lambda$) into itself. Hence it has a fixed-point, by Theorem 4.3.4. This means that the problem (*) has a unique solution. □

In this section we have outlined the approach to fixed-point theory which stems from order theory. We have therefore put an emphasis of fixed-point theorems which are genuinely order-theoretic in their character and attempted to prove their usefulness in various topological and analytical contexts. The discussion of these issues is far from being complete. The contemporary fixed-point theory is thorn between two conceptual frames. The first frame is rooted in the theory of order. The theorems presented in Section 4.2 are a sample of this approach. The second frame is anchored in topology and in the methods of topological convergence, especially in complete metric spaces. E.g. the original, direct proof of the Banach Fixed-Point Theorem resorts to a simple convergence argument. Other results, not mentioned in this chapter, as e.g. the Brouwer or Schauder Fixed-Point Theorems, belong to the widely understood theory of compact topological spaces. We may therefore speak of the order-oriented and of the topologically-oriented fixed-point theory, respectively.The Caristi-Kirka Fixed-Point Theorem can be viewed as a bridge theorem which (to some extent) fills a gap between these two conceptual frames. The proof of it uses Theorem 4.2.1, an equivalent form of the Axiom of Choice. It would be interesting to discover applications of the other theorems proved in Section 4.2, especially in the contexts in which Theorem 4.3.1 is involved. These theorems directly refer to inductive or chain-σ-complete posets. But no theorem determining general conditions under which the posets discussed in Theorem 4.3.1 are inductive is known in the literature so far.

4.4. Ordered action systems. Applications of fixed-point theorems.

In this and in the previous sections we are mainly concerned with applications of the above fixed-point theorems for relations to various, often diverse, domains of logic and mathematics as e.g. model theory or the theory of differential equations.

As a starter we show how to apply one of the above fixed-point theorems for relations – viz. Theorem 4.2.5 - to the proof of the downward Loewenheim -Skolem-Tarski Theorem. This theorem belongs to model theory. It states, roughly, that every infinite model ***A*** has an elementary submodel of any intermediate power between the cardinality of the language and the cardinality of ***A***. It would be also illuminating to see the proof of this theorem in the context of elementary action systems. We apply the standard model-theoretic notation. The notions we use such as a submodel, an elementary submodel etc. are not defined here. The reader is asked to see the standard textbooks on model theory to get acquainted with rudiments of this theory. A *language* is a set being the union of threes sets: a set of relational symbols (predicates), a set of function symbols, and a set of constant symbols. (Constant symbols are often viewed as nullary function symbols.) If L is a language, then $For(L)$ denotes the set of first-order formulas of L.

Theorem 4.4.1. (Downward Loewenheim-Skolem-Tarski Theorem).*Let L be a language and let α and β be cardinal numbers such that $|For(L)| \leq \beta \leq \alpha$.* Let ***A*** *be a model for L of cardinality α. Then **A** has an elementary submodel of cardinality β.*

*In fact, for any subset $X_0 \subseteq A$ of power $\leq \beta$, the model **A** has an elementary submodel of power β which contains X_0 as a subset of its universe.*

Proof. Let P be the family consisting of all subsets $X \subseteq A$ such that $|X| = \beta$ and $X_0 \subseteq X$. Evidently, P is non-empty because $|A| \geq \beta$. We have:

(A) *The family P, ordered by inclusion, is σ-complete*.

Indeed, for any non-empty countable chain ***C*** od subsets of A such that $|X| = \beta$ for all $X \in$ ***C***, the union $\cup$***C*** has cardinality β. Furhermore, the set **0** := X_0 is the least element of P. **0** is the supremum of the empty chain.

We define the following binary relation on the poset $(P, \subseteq)$: for X, $Y \in P$,

$X R Y$ iff $X \subseteq Y$ and for every formula $\Phi(x, x_1,..., x_n) \in For(L)$ and any sequence $a_1,..., a_n \in X$ (of length n) such that

$$\boldsymbol{A} \models (\exists x)\, \Phi\, [a_1,..., a_n]$$

there exists $b \in Y$ such that

$$\boldsymbol{A} \models \Phi\, [b, a_1,..., a_n].$$

$X R Y$ thus says that the set Y includes X and furthermore, for each formula $\Phi(x, x_1,..., x_n)$ and any n - tuple $a_1,..., a_n \in X$ satisfying $(\exists x)\, \Phi$ in $\boldsymbol{A}$, the set Y contains at least one $b \in A$ such that the $(n + 1)$-tuple $b, a_1,..., a_n$ satisfies $\Phi(x, x_1,..., x_n)$ in $\boldsymbol{A}$. The set Y may also contain some other elements of A but the cardinality of Y should not exceed β.

As $|For(L)| \leq \beta$, the crucial observation is that

(B) *The relation R is serial*, i.e., *for any $X \in P$ there exists $Y \in P$ such that $X R Y$*.

Furthermore, the definition of R gives that for any X, Y, X_1, X_2, $Y \in P$:

(C) *If $X R Y$ then $X \subseteq Y$.*

(D) *$X_1 \subseteq X_2$ and $X_2 R Y$ implies $X_1 R Y$.*

It follows from (B) and (C) that R is $\forall$-expansive. But, more interestingly,

(E) *The relation R is σ-continuous* (*in the stronger sense*).

We first check that R is monotone in $(P, \subseteq)$. Assume X, Y, $Z \in P$ so that $X \subseteq Y$ and $X R Z$. Evidently $Y \cup Z$ belongs to P. By (B) and (C) there exists a set W such that $Y \cup Z R W$ and $Y \cup Z \subseteq W$. As $Y \cup Z R W$, (D) gives that $Y R W$. Evidently $Z \subseteq W$. This proves monotonicity.

Suppose we are given two non-empty chains of elements of P,

$$Y_0 \subseteq Y_1 \subseteq ... \subseteq Y_n \subseteq Y_{n+1} \subseteq ... \text{ and } Z_0 \subseteq Z_1 \subseteq ... \subseteq Z_n \subseteq Z_{n+1} \subseteq ...$$

of types $\leq \omega$ such that $Y_n R Z_n$ for all n. It is clear that both the sets $\cup\{ Y_n : n \in \omega\}$ and $\cup \{Z_n : n \in \omega\}$ belong to P. We claim that

$$\cup\{ Y_n : n \in \omega\}\ R\ \cup\{Z_n : n \in \omega\}.$$

The fact that that $Y_n R Z_n$ for all n implies that $Y_n \subseteq Z_n$ for all n. Hence $\cup\{ Y_n : n \in \omega\} \subseteq \cup\{Z_n : n \in \omega\}$.

Denote $Y := \cup\{ Y_n : n \in \omega\}$ and $Z := \cup\{Z_n : n \in \omega\}$. Let $\Phi(x, x_1,..., x_k)$ be a formula in $For(L)$ and let $a_1,..., a_k$ be a sequence of elements of Y (of length k) such that

$$\boldsymbol{A} \models (\exists x)\ \Phi\ [a_1,\ldots, a_k].$$

There exists $n \in \omega$ such that $a_1,\ldots, a_k \in Y_n$. Since $Y_n\ R\ Z_n$, there exists $b \in Z_n$ such that

$$\boldsymbol{A} \models \Phi\ [b, a_1,\ldots, a_n].$$

Consequently, there exists $b \in Z$ such that

$$\boldsymbol{A} \models \Phi\ [b, a_1,\ldots, a_n].$$

As $\Phi\ (x, x_1,\ldots, x_k)$ is an arbitrary formula in *For* (L) and $a_1,\ldots, a_k$ are arbitrary elements of Y, this proves that $Y\ R\ Z$. So R is σ-continuous.

As R is serial, then trivially $R\ [\ \mathbf{0}] \neq \varnothing$. The system $(P, \subseteq, R)$ thus satisfies the hypotheses of Theorem 4.2.5. It follows that the relation R has a fixed-point in $(P, \subseteq)$, say B. It is then easy to verify that B is a universe of an elementary submodel $\boldsymbol{B}$ of $\boldsymbol{A}$. As B belongs to P, the cardinality of $\boldsymbol{B}$ is equal to β. □

The poset $(P, \subseteq)$, defined as in the proof of Theorem 4.4.1, is furnished with an infinite family of atomic actions, each action being associated with a formula of L. Indeed, for each formula $\Phi\ (x, x_1,\ldots, x_k)$ with al least one free variable x, we define the binary relation $A_\Phi \subseteq P \times P$ as follows: for X, $Y \in P$,

> $X\ A_\Phi\ Y$ iff $X \subseteq Y$ and for every sequence $a_1,\ldots, a_n \in X$ (of length n) such that
>
> $$\boldsymbol{A} \models (\exists x)\ \Phi\ [a_1,\ldots, a_n]$$
>
> there exists $b \in Y$ such that
>
> $$\boldsymbol{A} \models \Phi\ [b, a_1,\ldots, a_n].$$

A_Φ is called the *action of the formula* Φ.

It immediately follows from the definition of the relation R that

(F) $\qquad R = \bigcap \{A_\Phi : \Phi \in For\ (L)\}.$

(B) and (F) imply that for every $X \in P$, each action A_Φ is $\exists$-performable at X but it need not be $\forall$-performable. But (7) also implies that each realizable performance of whatever one action A_Φ is at the same time a realizable performance of all the remaining actions. Thus the agent who successfully performs one action, simultaneously performs all the other actions. We also not that each fixed-point of R is uniformly a fixed-point all the actions A_Φ, $\Phi \in For\ (L)$, i.e., X^* is a fixed-point of R iff it is a fixed-point of each relation A_Φ, $\Phi \in For\ (L)$.

As the ordered action system

$$(P, \subseteq, R, \{A_\Phi : \Phi \in For\ (L)\}\ )$$

is not normal, we can define another transition relation between states such that the resulting action system is normal. The fact that the system is normal ensures that every possible performance of an arbitrary action is realizable. We put:

$$R' := \cup \{A_\Phi : \Phi \in For(L)\}.$$

Then evidently, the system

$$(P, \subseteq, R', \{A_\Phi : \Phi \in For(L)\})$$

is normal, i.e, every atomic action A_Φ is $\forall$-performable in every state $X \in P$.

It is easy to see that R' has many fixed-points. In fact, the set of fixed-points of R' is the union of the set of fixed-points of the actions $A_\Phi : \Phi \in For(L)$. But not all fixed-points of R' qualify as solutions of the problem posed in Theorem 4.4.1, i.e., they need not even be submodels of $\boldsymbol{A}$. The right solutions are provided by states $X \in P$ which are uniformly fixed-points of all the actions A_Φ, $\Phi \in For(L)$.

We wish to present a number of remarks on ordered action systems. We begin with providing the official definition of such systems.

Definition 4.4.2. An *ordered elementary action system* is a quadruple

(1) $\boldsymbol{M} = (W, \leq, R, \boldsymbol{A})$,

where the reduct $(W, \leq)$ is a poset and the reduct $(W, R, \boldsymbol{A})$ is an elementary action system. □

Thus, in every ordered action system, the set W of states is assumed to be ordered. In practice, the system (1) is subject to further constraints. E.g., it will be often assumed that $(W, \leq)$ is chain-σ-complete and R satisfies some continuity or expansivity conditions with respect to $\leq$.

The main goal that underlines introducing ordered action systems on stage is the need of making a room for some infinite procedures in action. We call these procedures *approximations*. Mathematics admits such procedures. The process of computing consecutive decimal approximations of an irrational number is an example of such (potentially) infinite procedure. We write ”potentially” here because, in practice, this process halts at some stages because of the limitations imposed by the computer capacities and, last but not least, by finite time resources. But, theoretically, this procedure can be prolongated as long as one wishes. In this case, the members of W represent different approximations of a given irrational number. We may therefore identify each element of W with a pair consisting of a rational number r together with a measure of accuracy of the approximation provided by r. (It is assumed that such a measure is available.)

The states of (1) (i.e., the members of W) represent various possible phases of performing a certain task by the agents. It may be the process of building a house, sewing a dress etc. Intuitively, the order relation in (1) represents the degree of accomplishing this task.The fact that $u \leq v$ means that at the state v the process of accomplishing is more advanced than at u. E.g.it may mean that the state of the built house represented by v is more advanced than that represented by u.

Various courses of actions are of course possible. We may have the situation that $u \leq v_1$ and $u \leq v_2$, where v_1 and v_2 are even incomparable. This means that if the system is at the state u (i.e., u is a phase of realizing the task), several further courses of action, leading to more advanced stages of performing the task represented either by v_1 or by v_2, is conceivable. The fact that v_1 and v_2 are incomparable means that further strings of actiond undertaken at u may ramify. It may happen that there exists a later state w such that $v_1 \leq w$ and $v_2 \leq w$, which means that the ramified strings of actions converge, the agents return to the main track. But, generally, the task may not be unambigously determined and several options of completing the action are admissible. (E.g. the architect has designed various alternative versions of completing the house.) In this situation such a state w may not exist.

Definition 4.4.3. Let $\boldsymbol{M} = (W, \leq, R, \boldsymbol{A})$ be an ordered action system.

(1) $\boldsymbol{M}$ is said to be *chain-σ-complete* (*chain-complete* or *inductive*) if the poset $(W, \leq)$ is chain-σ-complete (inductive, respectively).

(2) An element $u^* \in W$ is called a *fixed-point* of $\boldsymbol{M}$ if $u^* R u^*$ and $u^* A u^*$ for every action $A \in \boldsymbol{A}$.

(2) An element $u^* \in W$ is called a *strong fixed-point* of $\boldsymbol{M}$ if it is a fixed-point of $\boldsymbol{M}$ and, furthermore, for every $w \in W$, if $u^* R w$, then $u^* \leq w$. □

The action system defined as in the proof of Theorem 4.4.1 is chain-σ-complete and the fixed-points investigated there are actually fixed-points of this system. It has another property: for every pair u, w of states, if $u R v$, then $u \leq v$. The same implication holds for every action A of the system: if $u A v$, then $u \leq v$, for all states u, w. This observation gives rise to the following definition:

Definition 4.4.4. An ordered action system $\boldsymbol{M} = (W, \leq, R, \boldsymbol{A})$ is called *constructive* if the relation R and the union $\cup\boldsymbol{A}$ are both subsets of $\leq$.

A system $\boldsymbol{M} = (W, \leq, R, \boldsymbol{A})$ is *destructive* if R and the union $\cup\boldsymbol{A}$ are both subsets of the dual order $\geq$. □

Destructive systems are thus constructive *a rebouir* – they are constructive in the sense of the reversed order $\geq$.

The system $(P, \subseteq, R, \{A_\Phi : \Phi \in For(L)\})$ is obviously constructive.

Constructive action systems should not be confused with the concept of constructive mathematics. The term "constructive mathematics" has a well established meaning which is not adequately captured by the above definition.

If an elementary action system $\boldsymbol{M} = (W, \leq, R, \boldsymbol{A})$ is constructive, then every fixed-point of $\boldsymbol{M}$ (if there is any) is a strong fixed-point. Indeed, assume that a^* is a fixed-point of $\boldsymbol{M}$. It follows that $a^* R a^*$. Assume furthermore that $a^* R w$ for some state w. As R is a subset of $\leq$, we have that $a^* \leq w$.

The proof of Theorem 4.4.1 shows that the following is true. In fact, Theorem 4.4.1 is an instantiation of the below theorem:

Theorem 4.4.5. *Let* $\boldsymbol{M} = (W, \leq, R, \boldsymbol{A})$ *be a chain-σ-complete action system such that* $R \subseteq \cap \boldsymbol{A}$*. If R is σ-continuous and* $R[\mathbf{0}] \neq \varnothing$*, then* **M** *has a fixed-point.*

Proof. Every fixed-point of R is a fixed-point of the system **M**. □

The following observation supplements Theorem 4.4.5.

Theorem 4.4.6. *Let* $\boldsymbol{M} = (W, \leq, R, \boldsymbol{A})$ *be a chain-complete action system such that* $R \subseteq \cap \boldsymbol{A}$*. Assume furthermore that R has the transfer property for upper bounds of chains and that the set* $R[\mathbf{0}]$ *is non-empty. Then* **M** *has a fixed-point.*

Proof. By Theorem 4.2.8, R has a fixed-point, say a^*. But the inclusion $R \subseteq \cap \boldsymbol{A}$ implies that a^* is a fixed-point of each action $A \in \boldsymbol{A}$. Hence a^*, and more generally, every fixed-point of R is a fixed-point of the system **M**. □

In ordered action systems $\boldsymbol{M} = (W, \leq, R, \boldsymbol{A})$, tasks for **M** usually take the form $(\{\mathbf{0}\}, \Psi)$, where $\mathbf{0}$ is the least element of $(W, \leq)$ and Ψ is a set of fixed-points of **M**. The process of carrying out the system **M** from the state **0** to a Ψ-state is not finitary since it usually involves various infinite limit passages. These passages are strictly linked with the process of approximating fixed-points of R by performing possibly infinite strings of consecutive actions of the system. In the simplest case, a fixed-point can be reached in ω steps.

The following notion proves convenient in the context of *ordered* action systems.

Definition 4.4.6. Let $\boldsymbol{M} = (W, \leq, R, \boldsymbol{A})$ be a chain-σ-complete elementary action system. The *ω-reach* of **M** is the binary relation $Z^{\omega}_{\boldsymbol{M}}$ on W defined as follows. For $u, v \in W$,

$Z^{\omega}_{\boldsymbol{M}}(u, v)$ iff there exist an ω-chain of states

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq u_{n+1} \leq \ldots$$

and a sequence of actions $A_0, A_1, \ldots, A_n, A_{n+1}, \ldots$ such that $u = u_0$, $v = \sup(\{u_n : n \in \omega\})$ and $u_n A_n, R\, u_{n+1}$, for all $n \in \omega$. □

$Z^{\omega}_{\boldsymbol{M}}(u, v)$ thus states that the state v is achiewed from the state u by means of performing an infinite string of actions A_n, $n \in \omega$, so that the outcomes of the actions form an increasing chain of states. The state v is the supremum of the resulting sequence of states.

The above chain $u_0 \leq u_1 \leq \ldots \leq u_n \leq u_{n+1} \leq \ldots$ has the property that $u_n R\, u_{n+1}$, for all $n \in \omega$. It need not hold that $u_m R \sup(\{u_n : n \in \omega\})$ for all (and even for some) m. This means that, generally, it is not possible to reach the state $v = \sup(\{u_n : n \in \omega\})$ from the states u_m by means of finitary procedures, i.e., by accomplishing a finite string of actions from **A**. This situation occurs in the system $(P, \subseteq, R, \{A_{\Phi} : \Phi \in For(L)\})$, where, generally, it is not possibe to produce an elementary submodel from the set **0** in a finite number of steps by applying the actions from $\{A_{\Phi} : \Phi \in For(L)\}$.

It is clear that in the system (P, $\subseteq$, R, $\{A_\Phi : \Phi \in For(L)\}$), every pair ($\mathbf{0}$, X) such that X is the universe of an elementary submodel of power β *of* the given model $\boldsymbol{A}$, belongs to the reach $Z^{\omega}_{\boldsymbol{M}}$.

4.5. Further fixed-point theorems. Ordered situational action systems.

In the previous paragraphs a few applications of fixed-point theorems for relations was presented, especially in the theory of models. It turns out, however, that model theory offers a variety of methods, which cannot be captured, at least directly, by the fixed-point theorems proved so far. In this section we discuss some of these model-theoretic constructions with the purpose of poviding a general, abstract framework for them based on the order-oriented fixed-point theory. The so called *back and forth method* is particularly useful in many branches of algebra and model theory. It dates back to the proof of the famous Cantor's theorem stating that any two countable linear dense orders without endpoints are isomorphic. The back and forth argument proved convenient in many branches of model theory, especially in the theory of saturated models. In order to find a plausible abstract formulation of the above construction, we shall discuss some modifications and refinements of the fixed-point theorems presented in § 4.2.

The notions we present in this section are certain modifications of the concepts we discussed in previous sections. These modifications are twofold in character. Firstly, some of the concepts defined in Section 4.2 are restricted to certain subsets of posets. Secondly, a certain weaker counterpart of the notions such as expansivity or monotonicity is formulated here. We called it conditional expansivity. We also define a certain restricted version of the σ-continuity of a relation.

We begin with the following remark. Let (P, $\leq$) be a poset. A mapping $\pi : P \to P$ is called *conditionally expansive* (or : *quasi-expansive*) if it satisfies the following quasi-identity :

$$(\forall x)\,(x \leq \pi(x) \to \pi(x) \leq \pi(\pi(x))).$$

Clearly, every expansive mapping is quasi-expansive. But it is also interesting to observe that every monotone mapping is quasi-expansive. Thus the notion of a conditionally expansive mapping is a comon generalization of the above two types of mappings associated with posets.

Given an element $a \in P$, we define :

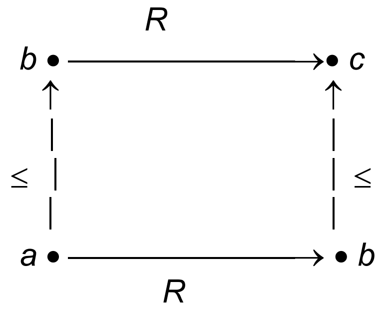
$$\pi^0(a) := a \text{ and } \pi^{n+1}(a) := \pi(\pi^n(a)), \text{ for all } n \geq 0.$$

We observe that if (P, $\leq$) has zero $\mathbf{0}$ and $\pi : P \to P$ is conditionally expansive, then the set $\{\pi^n(\mathbf{0}) : n \in \omega\}$ forms an increasing chain of type $\leq \omega$.

We will be mainly concerned with the following counterpart of the notion of quasi-expansivity for relations. Call a binary relation R on a poset $(P, \leq)$ *conditionally ∃-expansive* (or *quasi -∃-expansive*) if the following holds :

$$(\forall a, b \in P)[\, a \leq b \wedge a \; R \; b \rightarrow (\exists\, c \in P)\; b \; R \; c \wedge b \leq c \,]$$

(see the diagram below).



It is clear that every monotone relation, defined as in Section 4.2, is trivially quasi-∃-expansive. Furthermore, every ∃-expansive relation is quasi-∃-expansive. It is also evident that if R is the graph of a mapping $\pi : P \rightarrow P$, then the relation is quasi-∃-expansive in the above sense iff the mapping π is quasi-expansive.

Let $(P, \leq)$ be a chain-σ-complete poset. A relation $R \subseteq P \times P$ is said to be *conditionally σ-continuous* (or *quasi-σ-continuous*) if the following conditions hold :

(1) R is quasi-∃-expansive,

(2) For every countable chain C in $(P, \leq)$ of type $\leq \omega$ and for every monotone and expansive mapping $f : C \rightarrow P$, if $a \; R \; f(a)$ for all $a \in C$, then $\sup(C) \; R \; \sup(f[C])$.

We observe that in (2), due to the monotonicity of $f : C \rightarrow P$, the set $f[C]$ is a countable chain and therefore $\sup(f[C])$ exists. It is also clear that every σ-continuous relation is conditionally σ-continuous.

A mapping $\pi : P \rightarrow P$ is *conditionally σ-continuous* (or *quasi-σ-continuous*) if its graph has this property.

Theorem 4.5.1. *Let* $(P, \leq)$ *be a chain-σ-complete poset. A mapping* $\pi : P \rightarrow P$ *is conditionally σ-continuous iff it is quasi-expansive and for every chain* $a_0 \leq a_1 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots$ *of elements of* P *of type* $\leq \omega$, if $a_n \leq \pi(a_n)$ *and* $\pi(a_n) \leq \pi(a_{n+1})$ *for all* $n \in \omega$, *then* $\pi(\sup(\{a_n : n \in \omega\})) = \sup(\{\pi(a_n) : n \in \omega\})$.

The easy proof is omitted. □

It follows from these remarks that if $\pi : P \rightarrow P$ is conditionally σ-continuous, where $(P, \leq)$ is a chain-σ-complete poset, then the chain $\{\pi^n(\mathbf{0}) : n \in \omega\}$ is increasing and there holds the equality:

$$\pi(\sup(\{\pi^n(\mathbf{0}) : n \in \omega\})) = \sup(\{\pi(\pi^n(\mathbf{0})) : n \in \omega\}).$$

Consequently, the element $a^* := \sup(\{\pi^n(\mathbf{0}) : n \in \omega\})$ is a fixed-point of π.

We shall now modify a litle bit the above definitions. The modification consists in relating the above properties to certain selected subsets of posets.

Let $(P, \leq)$ be a chain-σ-complete poset and let P_0 be a subset of P. Furtherm-ore, let R be a binary relation on P. We say that R is *conditionally σ-continuous relative to* P_0 if the following conditions hold:

(3) R is conditionally ∃-expansive on P_0, i.e., for every pair $a, b \in P_0$ such that $a \leq b$ and $a\,R\,b$ there exists $c \in P_0$ such that $b\,R\,c$ and $b \leq c$.

(4) For every countable chain $C \subseteq P_0$ of type ω_0 and every monotone and expansive mapping $f: C \to P_0$, if $a\,R\,f(a)$ for all $a \in C$, then $\sup(C)\,R\,\sup(f[C])$.

(We note that $\sup(C)$ and $\sup(f[C])$ may *not* belong to P_0.)

A closer look at the proof of Theorem 4.2.5 reveals that its thesis can be reached under weaker assumptions:

Theorem 4.5.2. *Let* $(P, \leq)$ *be a chain-σ-complete poset and let* P_0 *be a subset of P. Assume that a relation* $R \subseteq P \times P$ *is conditionally σ-continuous relative to* P_0. *If* $\mathbf{0} \in P_0$ *and the set* $P_0 \cap R[\mathbf{0}]$ *is non-empty, then R has a fixed-point in P.*

Proof. We argue as in the proof of Theorem 4.2.7. $\mathbf{0}$ stands for the least element in $(P, \leq)$. Let a_0 be an arbitrary element of $P_0 \cap R[\mathbf{0}]$. As the elements $\mathbf{0}$ and a_0 belong to P_0 and $\mathbf{0}\,R\,a_0$ and $\mathbf{0} \leq a_0$, there exists, by the conditional monotonicity of R on P_0, an element $a_1 \in P_0$ such that $a_0\,R\,a_1$ and $a_0 \leq a_1$. Taking then the pair (a_0, a_1) and applying the conditional monotonicty of R on P_0, we see that there exists an element $a_2 \in P$ such that $a_1\,R\,a_2$ and $a_1 \leq a_2$. Taking in turn the pair (a_1, a_2), we see that there exists an element $a_3 \in P_0$ such that $a_2\,R\,a_3$ and $a_2 \leq a_3$, again by monotonicity. Continuing this pattern of argument, we define a countable chain: $\mathbf{0} \leq a_0 \leq a_1 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots$ of elements of P_0 such that $\mathbf{0}\,R\,a_0\,R\,a_1\,R \ldots R\,a_n\,R\,a_{n+1}\,R \ldots$. (see the figure following the proof of Theorem 4.2.5).

It readily follows from the conditional σ-continuity of R relative to P_0 that

$$\sup\{\mathbf{0}, a_0, a_1, \ldots, a_n, \ldots\}\ R\ \sup\{a_0, a_1, \ldots, a_n, \ldots\}.$$

Putting $a^* := \sup\{a_0, a_1, \ldots, a_n, \ldots\}$, we thus have that $a^*\,R\,a^*$. So a^* is a fixed-point of R. □

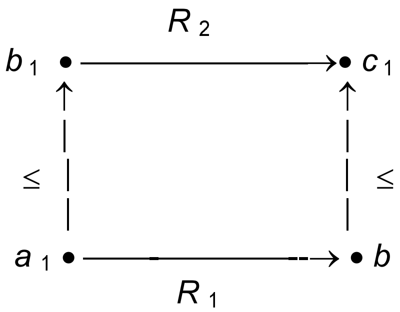
We discuss further modifications of the above definitions.

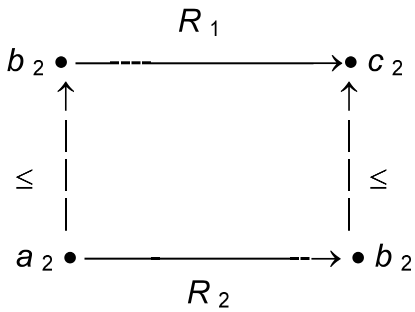
Let $(P, \leq)$ be a poset and let P_0 be a subset of P. Furthermore, let R_1 and R_2 be *two* binary relations on P. We say that R_1 and R_2 are *adjoint on* P_0 if the following two conditions hold:

(A1) $(\forall a_1, b_1 \in P_0)[\,a_1 \leq b_1 \wedge a_1\,R_1\,b_1 \to (\exists\, c_1 \in P_0)\ b_1\,R_2\,c_1 \wedge b_1 \leq c_1]$

(A2) $(\forall a_2, b_2 \in P_0)[\,a_2 \leq b_2 \wedge a_2\,R_2\,b_2 \to (\exists\, c_2 \in P_0)\ b_2\,R_1\,c_2 \wedge b_2 \leq c_2]$

(see the figures below).





The above two conditions can be neatly defined as one property on the direct products $P \times P$. Define the direct power $(P \times P, \leq)$ of the poset $(P, \leq)$ with the usual coordinate-wise definition of the order on $P \times P$. Thus

$$(a_1, a_2) \leq (b_1, b_2) \text{ iff } a_1 \leq b_1 \text{ and } a_2 \leq b_2 \text{ in } (P, \leq),$$

for all pairs $(a_1, a_2), (b_1, b_2) \in P \times P$.

Furthermore, if R_1 and R_2 are binary relations on P, then $R_1 \times R_2$ is the direct product of R_1 and R_2. Thus $R_1 \times R_2$ is a binary relation defined on $P \times P$ as follows: for $(a_1, a_2), (b_1, b_2) \in P \times P$,

$$(a_1, a_2)\ R_1 \times R_2\ (b_1, b_2) \text{ iff } a_1\ R_1\ b_1 \text{ and } a_2\ R_2\ b_2.$$

Properties (A1) and (A2) are expressed as a single property of the poset $(P \times P, \leq)$:

(A) For every two pairs $(a_1, a_2), (b_1, b_2) \in P_0 \times P_0$, if $(a_1, a_2) \leq (b_1, b_2)$ and $(a_1, a_2)\ R_1 \times R_2\ (b_1, b_2)$, then there exists a pair $(c_1, c_2) \in P_0 \times P_0$ such that $(b_1, b_2) \leq (c_1, c_2)$ and $(b_1, b_2)\ R_2 \times R_1\ (c_1, c_2)$.

Notice that in the antecedent of (A) the relation $R_1 \times R_2$ occurs. This relation is replaced in the succedent by $R_2 \times R_1$, the inverse of $R_1 \times R_2$. Thus (A) does *not* express the property of quasi-∃-expansivity of the relation $R_1 \times R_2$ on $P_0 \times P_0$.

The relations R_1 and R_2 are also called the *forth* and the *back relations on* P_0, respectively. The pair (R_1, R_2) is also called the *back and forth pair* (of relations) relative P_0.

Let $(P, \leq)$ be a chain-σ-complete poset and let P_0 be a subset of P. A pair (R_1, R_2) of binary relations on P is said to be *σ-continuously adjoint relative to* P_0 if R_1 and R_2 are adjoint on P_0 and, furthermore, for every chain $C = \{a_n : n \in \omega\}$ in P_0 of type $\leq \omega$ and for every monotone and expansive mapping $f : C \to P_0$ such that

$$a_{2n}\ R_1\ a_{f(2n)} \text{ and } a_{2n+1}\ R_2\ a_{f(2n+1)}, \text{ for all } n \in \omega,$$

there holds :

$$\sup(C)\ R_1 \sup(f[C]) \text{ and } \sup(C)\ R_2 \sup(f[C]).$$

(We note that the supremums $\sup(C)$ and $\sup(f[C])$ need not belong to P_0.)

An element $a^* \in P$ is called a *fixed-point* of the pair (R_1, R_2) if a^* is a fixed-point of both the relations R_1 and R_2, i.e., $a^* R_1 a^*$ and $a^* R_2 a^*$ hold.

Similarly to the case of back and forth pairs of relations, the above definitions can be expressed in terms of *one* relation only but defined on a poset having a more complicated set-theoretic structure than $(P, \leq)$, viz. the direct power $P \times P$. This remark implies that the conceptual framework within which we have studied fixed-points theorems thus far, i.e., the one provided by posets enriched with *one* binary relation (together with its inverse !), is sufficient, at least theoretically, for developing the results we prove in this section. However, for didactical and conceptual reasons, it is easier to work with the posets having *two* binary relations defined on them.

Theorem 4.5.3. *Let $(P, \leq)$ be a chain-σ-complete poset and let P_0 be a subset of P. Assume that a pair (R_1, R_2) of binary relations on P is σ-continuously adjoint relative to P_0. If $\mathbf{0} \in P_0$ and the set $P_0 \cap R_1[\mathbf{0}]$ is non-empty, then the pair (R_1, R_2) has a fixed-point in P.*

Proof. We suitably modify the proof of Theorem 4.5.2. We define a countable chain C (of type ω)

$$a_0 \leq a_1 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots$$

of elements of P_0. We put $a_0 := \mathbf{0}$. Let a_1 be an arbitrary element of $P_0 \cap R_1[\mathbf{0}]$. As $a_0, a_1 \in P_0$, $a_0 \leq a_1$ and $a_0 R_1 a_1$, there exists, by (A1), an element $a_2 \in P_0$ such that $a_1 \leq a_2$ and $a_1 R_2 a_2$. Taking then the pair a_1, a_2 and applying (A2), we see that there exists an element $a_3 \in P_0$ such that $a_2 \leq a_3$ and $a_2 R_1 a_3$. Then applying (A1) to the pair a_2, a_3, we find an element $a_4 \in P_0$ such that $a_3 \leq a_4$ and $a_3 R_2 a_4$ (see the figure below). Continuing, we define an increasing chain $C = \{a_n : n \in \omega\}$ in P_0 such that

$$a_0 R_1 a_1 R_2 a_2 R_1 a_3 R_2 a_4 \ldots a_{2n} R_1 a_{2n+1} R_2 a_{2n+2} \ldots$$

The mapping $f : C \to C$ defined by $f(a_n) := a_{n+1}$, for all $n \in \omega$, is expansive and monotone. Furthermore

$$a_{2n} R_1 f(a_{2n}) \quad \text{and} \quad a_{2n+1} R_2 f(a_{2n+1}), \text{ for all } n \in \omega.$$

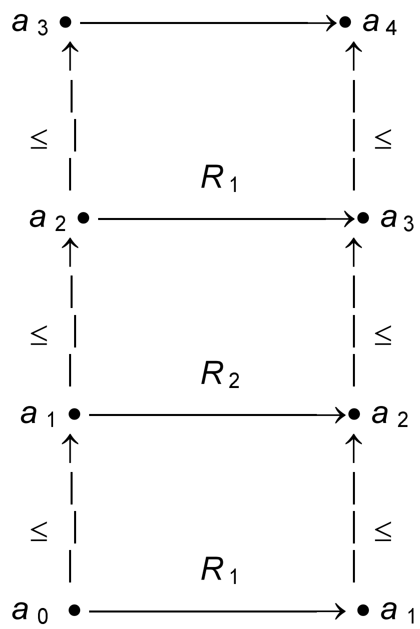
As the pair (R_1, R_2) is σ-continuously adjoint relative to P_0, we have that $\sup(C) R_1 \sup(f[C])$ and $\sup(C) R_2 \sup(f[C])$. Let $a^* := \sup(C)$. Evidently, $a^* = \sup(f[C])$. So $a^* R_1 a^*$ and $a^* R_2 a^*$. This concludes the proof of the theorem. □

As a simple (and somewhat trivial) application of Theorem 4.5.3 we give a proof of the following famous Cantor's theorem:

Theorem 4.5.4. *Every two countable linear and dense orders without end points are isomorphic.*

....

| R_2 |



Proof. Let $(X_1, \leq_1)$ and $(X_2, \leq_2)$ be two such orders. By a *partial isomorphism* (from $(X_1, \leq_1)$ to $(X_2, \leq_2)$) we mean any partial function $f : X_1 \to X_2$ such that f is injective on its domain $Dom(f)$ and, furthermore, for any elements $x, y \in Dom(f)$, $x \leq_1 y$ iff $f(x) \leq_1 f(y)$. A partial isomorphism $f : X_1 \to X_2$ is *finite* if its domain $Dom(f)$ is a finite set. **0** denotes the empty partial isomorphism. A (partial) isomorphism f is *total* if $Dom(f) = X_1$ and the co-domain $CDom(f)$ is equal to X_2.

Let P be the set of all partial isomorphisms from $(X_1, \leq_1)$ to $(X_2, \leq_2)$. P is partially ordered by the inclusion relation $\subseteq$ between partial isomorphisms. (Each partial isomorphism is a subset of the product $X_1 \times X_2$.) The poset $(P, \subseteq)$ is chain-σ-complete because the union of of any ω-chain of partial isomorphisms is easily seen to be a partial isomorphism. Furthermore, the empty isomorphism **0** is the least element in $(P, \subseteq)$.

We define two relations R_1 and R_2 on P. As X_1 and X_2 are countably infinite, we can write $X_1 = \{a_n : n \in \omega\}$ and $X_2 = \{b_n : n \in \omega\}$. Given partial isomorphisms f and g, we put :

> $f R_1 g$ iff *either* f is a total isomorphism and $g = f$ *or* f is a finite isomorphism and $g = f \cup \{(a_m, b_n)\}$, where
> (1) m is the smallest i such that $a_i \notin Dom(f)$,
> (2) n is the smallest j such that $b_j \notin CDom(f)$ and $f \cup \{(a_m, b_j)\}$ is a partial isomorphism.

(Note that the choice of n depends on the definition of m.)

> $f R_2 g$ iff *either* f is a total isomorphism and $g = f$ *or* f is a finite isomorphism and $g = f \cup \{(a_m, b_n)\}$, where
> (3) n is the smallest j such that $b_j \notin CDom(f)$,

(4) m is the smallest i such that $a_i \notin Dom(f)$ and $f \cup \{(a_i, b_n)\}$ is a partial isomorphism.

(Note that the choice of m depends on the definition of n.)

Let $P_0 \subseteq P$ be the set of all finite isomorphisms. Using the fact that the orders $(X_1, \leq_1)$ and $(X_2, \leq_2)$ are linear, dense, and without endponts, it is easy to verify that (R_1, R_2) is a back and forth pair of relations relative to P_0. But, more interestingly, the pair (R_1, R_2) is also σ-continuously adjoint relative to P_0. Evidently, the set $P_0 \cap R_1[\mathbf{0}]$ is non-empty. Hence, applying Theorem 4.5.2, we obtain that the pair (R_1, R_2) has a fixed-point in $(P, \subseteq)$, say f^*. It follows from the definition of R_1 and R_2 that f^* is actually a total isomorphism between $(X_1, \leq_1)$ and $(X_2, \leq_2)$. □

It should be observed that the relations R_1 and R_2 are partial functions with the same domain, viz. the set of finite partial isomorphisms plus the set of total isomorphisms. E.g. partial isomorphisms f such that either $Dom(f) = X_1$ and $CDom(f) \neq X_2$ or $Dom(f) \neq X_1$ and $CDom(f) = X_2$ do not belong to the common domain of R_1 and R_2. Furthermore, the existence of a total isomorphism between $(X_1, \leq_1)$ and $(X_2, \leq_2)$ is already implicit in the fact that the pair (R_1, R_2) is σ-continuously adjoint relative to P_0, as can be easily checked. Thus, the above proof does not give a new insight into the original Cantor's proof. The significance of the above proof consists in the fact that it is uniformly formulated in the general, abstract framework provided by Theorem 4.5.3.

The above example gives rise to the construction of a certain simple situational action system. The set of states of the system coincides with the set P of partial isomorphisms between $(X_1, \leq_1)$ and $(X_2, \leq_2)$. The system is equipped with two elementary actions: R_1 and R_2, the forth and the back actions. By way of analogy with game of chess, it is assumed that the system has two agents: BACK and FORTH. The agent FORTH performs the action R_1 while BACK performs R_2. The agents are highly cooperative – they obey to the rule that their actions are performed alternately. Furthermore, they are succesful in action – in the infinite they are able to define a required total isomorphism between the posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$.

A *possible situation* is a pair $s = (u, \alpha)$, where u is a partial isomorphism and α is an agent, i.e., $\alpha \in$ {BACK, FORTH}. u is the (unique) state of the system corresponding to the situation s. This state is marked as $f(s)$.The situation s is read : "The agent α performs his action in the state u". The situation ($\mathbf{0}$, FORTH) is called *initial*. (We recall that $\mathbf{0}$ is the empty partial isomorphism.)

Let S be the set of possible situations. The *transition relation Tr* between situations is defned as follows. For any situations s and t,

$s\ Tr\ t$ iff *either* $s = (u, \text{FORTH})$, $t = (v, \text{BACK})$ and $u\ R_1\ v$

or $s = (u, \text{BACK})$, $t = (v, \text{FORTH})$ and $u\ R_2\ v$.

We thus see that if $s\ Tr\ t$, then either $f(s)\ R_1\ f(t)$ or $f(s)\ R_1\ f(t)$. The union $R_1 \cup R_2$ is called the *transition relation between states*. Note that Tr is a (partial) function.

The set S_a of *actual situations* is defined as follows. The initial situation $s_0 := (\mathbf{0}, \text{FORTH})$ is actual. If s_n is actual, then there exists a (unique) situation s_{n+1} such that $s_n\ Tr\ s_{n+1}$. Then s_{n+1} is actual as well. Furthermore, it is assumed that the situations of the form (g, FORTH) and (g, BACK) are actual, where g is the total isomorphism being the union of the finite isomorphisms $f(s_n)$, $n \in \omega$. (g, FORTH) and (g, BACK) are *terminal* actual situations.

The system $\boldsymbol{M}^s := (P, R_1 \cup R_2, \{R_1, R_2\}, S, Tr, f, S_a\}$ is thus a situational action system. $\boldsymbol{M}^s$ is called the *system of the back and forth actions.* . $\boldsymbol{M}^s$ is ordered in the sense that its set of states P is ordered by $\leq$. The extended structure $(P, \leq, R_1 \cup R_2, \{R_1, R_2\}, S, Tr, f, S_a)$ is an example of the so called ordered situational action systems. (See the remarks placed at the end of this section.)

We recall that an elementary action system $\boldsymbol{M} = (W, R, \boldsymbol{A})$ is *constructive* if the relation R and each of the actions $A \in \boldsymbol{A}$ are subsets of $\leq$. If $\boldsymbol{M}$ is constructive, the relation R and the actions $A \in \boldsymbol{A}$ need not be $\forall$-expansive because they are not assumed to be serial. By the same reason, the above relations need not be $\exists$-expansive. In the above example, the reduct $\boldsymbol{M} := (P, R_1 \cup R_2, \{R_1, R_2\})$ of $\boldsymbol{M}^s$ is clearly a constructive and chain - σ- complete elementary action system.

A relation «, defined on a set S, is called a *quasi-order* if « is reflexive and transitive. It is well-known that if « is a quasi-order, then the binary relation ρ defined on S as follows

(5) $\qquad a \rho b$ iff a « b and b « a,

($a, b \in S$) is an equivalence relation on S. Let $[a]_\rho$ stand for the equivalence class of the element a relative to ρ:

$$[a]_\rho := \{x \in S : a \rho x\}.$$

The quotient set

$$S/\rho := \{[a]_\rho : a \in S\}$$

is then partially ordered by the relation $\leq$, where

$$[a]_\rho \leq [b]_\rho \text{ iff } a \text{ « } b$$

($a, b \in S$).

Definition 4.5.5. Let $\boldsymbol{M}^s = (W, R, \boldsymbol{A}, S, Tr, f, S_a\}$ be a situational action system. The system $\boldsymbol{M}^s$ is *ordered* if its reduct $\boldsymbol{M} := (W, R, \boldsymbol{A})$ is an ordered action system, i.e., the set W is endowed with a fixed partial order $\leq$. □

In what follows ordered situatinal action systems are marked as

(6) $\qquad \boldsymbol{M}^s = (W, \leq, R, \boldsymbol{A}, S, Tr, f, S_a)$,

thus explicitly indicating the order relation $\leq$ on the set of states W.

Given an ordered situational action system (6), we see that the partial order $\leq$ can be lifted to a quasi-order on the set S of possible situations. For any situations a, $b \in S$, we define:

(7) $\qquad a \ll b$ iff $f(a) \leq f(b)$,

i.e., the state $f(a)$ corresponding to a is equal or less than the state $f(b)$ corresponding to b. An easy verification that $\ll$ is indeed a quasi-order is left for the reader.

The following observation is obvious:

Proposition 4.5.6. *Let* $\boldsymbol{M}^s = (W, \leq, R, \boldsymbol{A}, S, Tr, f, S_a)$ *be an ordered situational system. Define the quasi-order* $\ll$ *and the equivalence relation* ρ *on the set S of situations as in* (7) *and* (5). *Then the following conditions hold, for any* a, $b \in S$:

(8) $\qquad [a]_\rho = [b]_\rho$ *iff* $f(a) = f(b)$.

(9) $\qquad [a]_\rho \leq [b]_\rho$ *iff* $f(a) \leq f(b)$.

(10) $\qquad$ *The mapping* f^*, *which to each equivalence class* $[a]_\rho$ *assigns the state* $f(a)$, *is well-defined. Furthermore,* f^* *is an isomorphism between the posets* $(S/\rho, \leq)$ *and* $(W, \leq)$. □

An ordered situational action system $\boldsymbol{M}^s = (W, \leq, R, \boldsymbol{A}, S, Tr, f, S_a)$ is *chain-σ-complete* (*chain-complete*) if its reduct $(W, \leq)$ is a chain-σ-complete poset (inductive poset, respectively).

In the context of ordered situational systems, it is useful to have another infinite notion of a reach, more closely related to the properties of the set S of situations of the system (cf. Definition 4.4.6).

Let $\boldsymbol{M}^s = (W, \leq, R, \boldsymbol{A}, S, Tr, f, S_a)$ be a chain-σ-complete situational system. By the *situational* ω*-reach of the system* $\boldsymbol{M}^s$ we mean a binary relation SZ^ω on the set S defined as follows. For any $a, b \in S$,

$SZ^\omega(a, b)$ iff there exists a sequence

$$a_0, a_1, \ldots, a_n, a_{n+1}, \ldots$$

of situations of S and a sequence $A_0, A_1, \ldots, A_n, A_{n+1}, \ldots$ of actions of $\boldsymbol{A}$ such that

(i) $\quad a_n \, Tr \, a_{n+1}$ and $f(a_n) \leq f(a_{n+1})$, for all $n \in \omega$,

(ii) $\quad a_n A_n a_{n+1}$, for all $n \in \omega$,

(iii) $\quad a_0 = a$ and $f(b) = \sup(\{f(a_n) : n \in \omega\})$.

Note that (i) can be formulated as:

(i) * $a_n \ll a_{n+1}$, for all $n \in \omega$.

Since a_n Tr a_{n+1} implies that $f(a_n)$ R $f(a_{n+1})$, for all $n \in \omega$, it is easy to see that if $SZ^{\omega}(a, b)$, then $Z^{\omega}_{\boldsymbol{M}}(f(a), f(b))$ in the sense of Definition 4.4.6, but not conversely. We recall that $Z^{\omega}_{\boldsymbol{M}}$ is the ω-reach of the elementary system $\boldsymbol{M} := (W, \leq, R, \boldsymbol{A})$, being a reduct of $\boldsymbol{M}^s$.

For example, in the situational system $\boldsymbol{M}^s =$ $(P, R_1 \cup R_2, \{R_1, R_2\}, S, Tr, f, S_a\}$ of the back and forth actions, defined as above, there exists a unique total isomorphism g between the posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$ such that SZ^{ω} ((**0**, FORTH), (g, FORTH)) and SZ^{ω}((**0**, FORTH), (g, BACK)).

4.6. An algebraic treatment of infinitistic definitions.

Calculus uses the notion of a limit. The methods and the definitions calculus provides are based on some genuinely limit procedures. (It is well-known that calculus is traced back to the method of exhaustion.) They can be described as certain approximations in ordered structures. In this section we present a cluster of remarks which, we believe, shed more light on this issue.

Algebraic posets

Let $(P, \leq)$ be a directed-complete poset. An element $x \in P$ is *compact* if, whenever D is a directed subset of P and $x \leq \sup(D)$, there is an element $d \in D$ such that $x \leq d$. We let $K(P)$ denote the set of compact elements of $(P, \leq)$.

As every directed-complete poset contains zero **0**, we note that **0** is *not* compact because it is the supremum of the empty directed subset.

Definition 4.6.1. A poset $(P, \leq)$ is said to be *algebraic* if it satisfies the following conditions:

(1) $(P, \leq)$ is directed-complete ;

(2) For every $a \in P$, the set $K(a) := \{x \in K(P) : x \leq a\}$ is directed and $\sup(K(a)) = a$. □

In other words, in an algebraic poset, each element is a directed limit of its ”finite”, i.e., compact approximations. The notion of an algebraic poset thus generalizes the well-known concept of an algebraic lattice.

As every algebraic poset has the least element **0**, we see that $K(\mathbf{0}) = \emptyset$.

If $(P, \leq)$ is algebraic and $K(P)$ is countable, then $(P, \leq)$ is called a *domain.*

Definition 4.6.2. Let $(P_0, \leq)$ be a poset. A subset $I \subseteq P_0$ is called an *ideal* of $(P_0, \leq)$ if the following conditions hold, for all $a, b \in P_0$:

(i 1) If $a, b \in I$, then there exists $c \in I$ such that $a \leq c$ and $b \leq c$. (I.e., I is a directed subset of $(P_0, \leq)$.)

(i 2) If $a \in I$ and $b \leq a$, then $b \in I$. □

The empty subset is an ideal. It is the smallest ideal. If $(P_0, \leq)$ has the least element **0**, then $\{ \mathbf{0} \}$ is the least non-empty ideal of $(P_0, \leq)$. If $(P_0, \leq)$ is directed, then P_0 is the largest ideal. If $\boldsymbol{X}$ is a directed family of ideals of $(P_0, \leq)$, then the union $\cup\boldsymbol{X}$ is also an ideal. We note that the intersection of two ideals need not be an ideal. For let $P_0 = \{a, b, c, d\}$ and declare: $a < c$, $a < d$, $b < c$, $b < d$. Both the sets $\{a, b, c\}$ and $\{a, b, d\}$ are ideals but their intersection is not an ideal.

For every $a \in P_0$, the set

$$(a] := \{ x \in P_0 : x \leq a \}$$

is an ideal. $(a]$ is called the *principal ideal* generated by a.

Let $Id(P_0, \leq)$ be the set of all ideals of $(P_0, \leq)$. If the order $\leq$ on P_0 is clear from context, the set $Id(P_0, \leq)$ is briefly denoted by $Id(P_0)$.

Clearly, the family $Id(P_0)$, ordered by inclusion, is a poset.

Here is a number of simple observations about the posets of ideals.

Theorem 4.6.3. *Let $(P_0, \leq)$ be an arbitrary poset. The poset $(Id(P_0), \subseteq)$ of ideals of $(P_0, \leq)$ is algebraic. Furthermore the subposet $(K(Id(P_0)), \subseteq)$ of compact elements of the poset $(Id(P_0), \subseteq)$ is isomorphic with $(P_0, \leq)$.*

Proof. The poset $(Id(P_0), \subseteq)$ is directed-complete. Indeed, if $(X, \subseteq)$ is a directed family of ideals, then the union $\cup X$ is also an ideal of $(P_0, \leq)$. Clearly, $\cup X$ is the supremum of the family X in the sense of $(Id(P_0), \subseteq)$.

We prove that the family $\{ (a] : a \in P_0 \}$ of principal ideals of $(P_0, \leq)$ coincides with the family $K(Id(P_0))$. Let $a \in P_0$ and let X be a directed family of ideals such that $(a] \subseteq \cup X$. It follows that there exists an ideal $I \in X$ such that $a \in I$. Hence $(a] \subseteq I$. This shows that that $(a]$ is compact in the poset $(Id(P_0), \subseteq)$.

To prove the converse, we notice that $I = \cup\{ (a] : a \in I \}$ for every ideal I. Thus every ideal is a supremum of compact filters. Furthermore, the family $\{ (a] : a \in I \}$ is directed. Now, if a non-empty ideal I is a compact element in $(Id(P_0), \subseteq)$, it follows from the above equality that $I \subseteq (a]$ for some $a \in I$. Hence $I = (a]$.

The above facts prove that the poset $(Id(P_0), \subseteq)$ is algebraic.

It is clear that the mapping $\varphi : P_0 \to Id(P_0)$ given by

$$\varphi(a) := (a], \quad a \in P_0,$$

establishes an isomorphism between the posets $(P_0, \leq)$ and $(K(Id(P_0)), \subseteq)$. □

Note. The mapping φ, defined as above, need *not* preserve supremums of directed subsets of $(P_0, \leq)$, that is, if D is a directed subset of $(P_0, \leq)$ such that $\sup(D)$ exists in $(P_0, \leq)$, then clearly $\varphi[D]$ is a directed subset of $(K(Id(P_0)), \subseteq)$ but $\varphi(\sup(D))$ need not be equal to $\sup(\varphi[D])$. (The last supremum is taken in $K(Id(P_0))$.) □

The order structure of any algebraic poset is fully determined by the order structure of the subposet of its compact elements. This is the content of the following theorem:

Theorem 4.6.4. *Let* $(P, \leq)$ *and* $(Q, \leq)$ *be algebraic posets.* $(P, \leq)$ *and* $(Q, \leq)$ *are isomorphic iff the posets* $(K(P), \leq)$ *and* $(K(Q), \leq)$ *of their compact elements are isomorphic.*

The proof is based on several simple lemmas.

Lemma 4.6.5. *Let* $f : (P_0, \leq) \cong (Q_0, \leq)$ *be an isomorphism between posets. The mapping* f^* *given by the formula*

$$f^*(J) := f^{-1}(J), \text{ for all } J \in Id(Q_0),$$

is an isomorphism between the posets $(Id(Q_0), \subseteq)$ *and* $(Id(P_0), \subseteq)$*, symbolically,*

$$f^* : (Id(Q_0), \subseteq) \cong (Id(P_0), \subseteq).$$

The easy proof is omitted. □

Lemma 4.6.6. *Let* $(P, \leq)$ *be an algebraic poset. Then* $(P, \leq)$ *is isomorphic with the poset* $(Id(K(P), \subseteq)$ *of ideals of the poset* $(K(P), \leq)$ *of compact elements of* $(P, \leq)$.

Proof of the lemma. For every $a \in P$ define $K(a)$ as above, i.e.,

$$K(a) := \{ x \in K(P) : x \leq a \}.$$

Since $(P, \leq)$ is algebraic, $K(a)$ is directed and hence it is an ideal of $(K(P), \leq)$.

Claim 1. *Let* I *be an ideal of* $(K(P), \leq)$. *Then* $I = K(a)$ *for some* $a \in P$.

Proof of the claim. As I is a directed subset of $(K(P), \leq)$, and hence of $(P, \leq)$, $\sup(I)$ exists in $(P, \leq)$. (Note that if I is empty, $\sup(I) = \mathbf{0}$.) We define: $a := \sup(I)$. Evidently, $I \subseteq K(a)$ because I consists of compact elements and every element of I is equal or smaller than a. Now, let $x \in K(a)$. So x is compact and $x \leq a = \sup(I)$. Hence $x \leq i$ for some $i \in I$ by compactness. Since I is an ideal, we therefore have that $x \in I$. So $K(a) \subseteq I$. Consequently, $I = K(a)$. This proves the claim.

Claim 2. *Let* a, $b \in P$. *Then* $K(a) \subseteq K(b)$ *iff* $a \leq b$.

Proof of the claim. Use the fact that $a = \sup(K(a))$ and $b = \sup(K(b))$.

To prove the lemma, we define the mapping $h : P \to Id(K(P))$ by:

$$h(a) := K(a), \text{ for all } a \in P.$$

Clearly, h is well-defined. By Claim 1, h is surjective. By Claim 2, h is an isomorphism and hence it is one-to-one. □

Lemma 4.6.7. *Let* $(P, \leq)$ and $(Q, \leq)$ *be algebraic posets. If* $(P, \leq)$ *and* $(Q, \leq)$ *are isomorphic*, *then so are the posets* $(K(P), \leq)$ *and* $(K(Q), \leq)$.

Proof of the lemma. Let $f : (P, \leq) \cong (Q, \leq)$ be an isomorphism. It suffices to prove that f maps $K(P)$ onto $K(Q)$. Thus we need to show that, for every $a \in P$, $a \in K(P)$ iff $f(a) \in K(Q)$. A tedious verification of this condition is left to the reader. □

We now pass to the proof of Theorem 4.6.4.

($\Leftarrow$). We assume that

$$(K(P), \leq) \cong (K(Q), \leq).$$

Lemma 4.6.5 then implies that

$$(Id(K(P)), \subseteq) \cong (Id(K(Q)), \subseteq).$$

In turn, Lemma 4.6.6 gives that

$$(P, \leq) \cong (Id(K(P)), \subseteq) \text{ and } (Q, \leq) \cong (Id(K(Q)), \subseteq).$$

Consequently, $(P, \leq) \cong (Q, \leq)$.

($\Rightarrow$). This is the content of Lemma 4.6.7.

The theorem has been proved. □

The following result follows from Theorems 4.6.3 – 4.6.4:

Theorenm 4.6.8. *Let* $(P_0, \leq)$ *be an arbitrary poset. There exists a unique* (*up to isomorphism*) *poset* $(P, \leq)$ *with the following properties* :

(1) $(P, \leq)$ *is algebraic*.

(2) *The poset* $K(P)$ *of compact elements of* $(P, \leq)$ *is isomorphic with* $(P_0, \leq)$.

Proof. Indeed, In view of Theorem 4.6.3, the poset $(Id(P_0), \subseteq)$ of ideals of

$(P_0, \leq)$ satisfies (1) and (2). In view of Theorem 4.6.4, there is only one (up to isomorphism) poset $(P, \leq)$ which satisfies (1) – (2). □

The unique algebraic poset $(P, \leq)$ satisfying the above conditions (1) – (2) is called the *algebraic completion* of the poset $(P_0, \leq)$. In view of Theorems 4.6.3 – 4.6.4, the poset $(Id(P_0), \subseteq)$ of ideals of $(P_0, \leq)$ is *the* algebraic completion of $(P_0, \leq)$.

The following observation is crucial :

Theorem 4.6.9. *Let* $(P, \leq)$ *be an algebraic poset and let* $(Q, \leq)$ *be a directed-complete poset. Furthermore, let* $F_0 : K(P) \to Q$ *be a monotone function defined on the set of compact elements of* $(P, \leq)$. *Then there exists an order continuous function* $F : P \to Q$, *which is an extension of* F_0. *Furthermore,* F *is unique up to the zero element, that is, any other continuous extension of* F_0 *agrees with* F *on the set* $P - \{\mathbf{0}\}$.

Proof. For every $a \in P$, we define $K(a)$ as before, i.e.,

$$K(a) := \{ x \in K(P) : x \leq a \}.$$

Since the poset $(P, \leq)$ is algebraic, the set $K(a)$ is directed. Furthermore, for any a, $b \in P$, $a \leq b$ if and only if $K(a) \subseteq K(b)$.

For every $a \in P$, we define :

$$F(a) := \sup(\{ F_0(x) : x \in K(a)\}).$$

F is a well-defined mapping from the poset $(P, \leq)$ into $(Q, \leq)$. Indeed, since F_0 is monotone, $\{F_0(x) : x \in K(a)\}$ is a directed subset of $(Q, \leq)$, and hence $\sup(\{F_0(x) : x \in K(a)\})$ exists in $(Q, \leq)$, for every $a \in P$. The inequality $a \leq b$ then implies that $F(a) \leq F(b)$, for all $a, b \in P$. This means that F is monotone.

We also note that if $a = \mathbf{0}$ and hence $K(\mathbf{0})$ is empty, then $F(\mathbf{0})$ is the zero of $(Q, \leq)$.

We claim that $F(x) = F_0(x)$, for all $x \in K(P)$. Indeed, for every $x \in K(P)$ we have that $F(x) = \sup(\{ F_0(y) : y \in K(x)\}) \leq F_0(x)$, by the monotonicity of F_0. On the other hand, if x is compact, then $F_0(x) \in \{F_0(y) : y \in K(x)\}$. Hence $F_0(x) \leq \sup(\{ F_0(y) : y \in K(x)\}) = F(x)$. Consequently, $F(x) = F_0(x)$ whenever $x \in K(P)$.

We prove that F is order continuous. Suppose D is a non-empty directed subset of $(P, \leq)$. The union $K := \cup \{ K(d) : d \in D \}$ is also directed and $K = K(\sup(D))$. The definition of F then gives that

$$F(\sup(D)) = \sup(\{ F_0(x) : x \in K(\sup(D))\}) = \sup(\{ F_0(x) : x \in K \}) =$$

$$\sup(\{ F_0(x) : x \in \cup \{ K(d) : d \in D \} \}) =$$

$$\sup(\{ \sup(\{ F_0(x) : x \in K(d) \}) : d \in D \}) =$$

$$\sup(\{ F(d) : d \in D\}).$$

This shows that F is continuous.

To prove uniqueness, suppose F_1 and F_2 are continuous extensions of F_0 onto P. Let $a \in P$, $a \neq \mathbf{0}$. Then $K(a)$ is non-empty. Hence, by continuity, $F_1(a) = F_1(\sup(K(a))) = \sup(\{ F_1(x) : x \in K(a) \}) = \sup(\{ F_0(x) : x \in K(a) \}) = \sup(\{ F_2(x) : x \in K(a) \}) = F_2(\sup(K(a)) = F_2(a)$. □

Integral posets

The poset $(P, \subseteq)$ of partial isomorphisms, defined as in the proof of Theorem 4.5.3, is a semantic domain. $(P, \subseteq)$ is evidently directed-complete. The set $K(P)$ of its compact elements coincides with the set P_0 of finite partial isomorphisms. Note, however, that P_0 itself is *not* directed because, generally, not every two finite isomorphisms can be combined to form a partial isomorphism. P_0 is countable because the sets X_1 and X_2 are countable.

The algebraic poset $(P, \subseteq)$ of partial isomorphisms does not have a greatest element but it possesses maximal elements. In this section we examine algebraic posets which possess unit element **1**, the largest element of the poset. Clearly, in each such a poset, $K(P) = \{ x \in K(P) : x \leq \mathbf{1}\}$ and $\sup(K(P)) = \mathbf{1}$.

Definition 4.6.10. An algebraic poset $(P, \leq)$ is called *integral* if the set $K(P)$ of its compact elements is directed. □

Proposition 4.6.11. *Let* $(P, \leq)$ *be an algebraic poset. The poset* $(P, \leq)$ *is integral if and only if it it has the unit element* **1**.

Proof. (⇒). Indeed, if $(P, \leq)$ is integral, then $\sup(K(P))$ exists in $(P, \leq)$ because $K(P)$ is a directed subset. But due to the fact that *every* element of P is a supremum of a set of compact elements, it follows that $\sup(K(P))$ is the greatest element in $(P, \leq)$. So $\mathbf{1} = \sup(K(P))$.

(⇐). Conversely, if $(P, \leq)$ has unit **1**, then the set $\{ x \in K(P) : x \leq \mathbf{1}\}$ is directed and $\mathbf{1} = \sup(\{ x \in K(P) : x \leq \mathbf{1}\})$, by the algebraicity of $(P, \leq)$. But evidently, $\{ x \in K(P) : x \leq \mathbf{1}\} = K(P)$. Hence $K(P)$ is directed. □

Integral posets will be marked as $(P, \leq, \mathbf{1})$, thus indicating the existence of the greatest element in them.

The following result follows from Theorem 4.6.8..

Theorem 4.6.12. *Let* $(P_0, \leq)$ *be a directed poset. There exists a unique (up to isomorphism) poset* $(P, \leq, \mathbf{1})$ *with the following properties* :

(1) $(P, \leq, \mathbf{1})$ *is integral*.

(3) *The poset* $K(P)$ *of compact elements of* $(P, \leq)$ *is isomorphic with*

$(P_0, \leq)$.

Proof. It suffices to notice that the unique algebraic completion of $(P_0, \leq)$ is integral. But the poset $(Id(P_0), \subseteq)$ of ideals of the poset $(P_0, \leq)$ *is* the algebraic completion of $(P_0, \leq)$. Furthermore, as $(P_0, \leq)$ is directed, P_0 itself is the greatest ideal of of $(P_0, \leq)$. So $(Id(P_0), \subseteq)$ has the unit element. □

Examples.

(1). The ordinals are defined in the standard way *à la* von Neumann. Let α be an ordinal number. α is well-ordered by the inclusion relation. If α is not a non-zero limit ordinal, then the algebraic completion of $(\alpha, \subseteq)$ is isomorphic with $(\alpha, \subseteq)$. If α is a non-zero limit ordinal, then the algebraic completion of $(\alpha, \subseteq)$ is isomorphic with its successor $(\alpha \cup \{\alpha\}, \subseteq)$.

ω is the least non-zero limit ordinal. The elements of ω are called *natural numbers* . Thus $\omega = \{0, 1, 2, ...\}$, where $0 := \varnothing$ and $n + 1 := \{0, 1, ..., n\}$ for all n. It follows from the above remark that the algebraic completion of $(\omega, \subseteq)$ is isomorphic with $(\omega \cup \{\omega\}, \subseteq)$.

(2). $(Q, \leq)$ denotes the set of rational numbers with the usual ordering. As this poset is linear, it is evidently directed. It is easy to describe the set of ideals of $(Q, \leq)$. Since the order $(Q, \leq)$ is linear, a set I is an ideal of $(Q, \leq)$ iff, for any $a, b \in Q$, $a \leq b$ and $b \in I$ imply $a \in I$. The empty set is the least ideal and Q - the lagest one. The empty ideal **0** is customarily denoted by $-\infty$. The largest ideal **1**, which is equal to$Q$, is denoted by $+\infty$. Otherwise, an ideal I is called *prope*r.

Given a proper ideal I of $(Q, \leq)$, two cases are posiible:

Case 1. I has a greatest element.

In this case $I = (q]$ for some $q \in Q$. The ideals of this form are exactly the compact members of the integral poset $(Id(Q), \subseteq, \mathbf{1})$.

Case 2. I has no greatest element.

If Case 2 holds, the proper ideal I is called a *schnitt*.

Following Dedekind, the set of real numbers is identified with the collection ***R*** of all schnitts. The set Q of rational numbers is then identified with the collection of schnitts $\{(q) : q \in Q\}$, where $(q) := \{p \in Q : p < q\}$. However, in our approach the elements of the poset $(Q, \leq)$ are identified with compact ideals of its algebraic completion $(Id(Q), \subseteq, \mathbf{1})$. Consequently, each rational number q is identified with the principal ideal $(q]$.

We thus see that the poset $(Id(Q), \subseteq, \mathbf{1})$, the algebraic completion of $(Q, \leq)$, is a domain which is *larger* than the set ***R*** of real numbers with the standard ordering enriched with the elements $-\infty$ and $+\infty$. Indeed, apart from the standard irrational numbers, identified with the collection of schnitts which are *not* of the form (q), where q is a rational, and the collection of rational numbers, identified with the set of compact ideals $(q]$, where $q \in Q$, the poset $(Id(Q), \subseteq)$ also contains countably many ideals (q), $q \in Q$. In the order structure of $(Id(Q), \subseteq)$, each element (q) is smaller than the rational $(q]$, but it is infinitesimally close to $(q]$ in the sense that (q) is greater than any rational $(p]$ strictly smaller than $(q]$.

(3). $(\boldsymbol{R}, \leq)$ denotes the set of real numbers with the usual ordering. The poset $(\boldsymbol{R}, \leq)$ is directed but not directed-complete because it does not have a least element. The extended poset $(\boldsymbol{R} \cup \{-\infty, +\infty\}, \leq)$ is directed-complete but it is *not* algebraically closed. We shall describe the elements of the algebraic completion $(Id(\boldsymbol{R}), \subseteq, \mathbf{1})$ of $(\boldsymbol{R}, \leq)$. $\mathbf{0} := \varnothing\ (= -\infty)$ is the least ideal. $\mathbf{1} := \boldsymbol{R}\ (=+\infty)$ is the largest ideal. The compact ideals of $(\boldsymbol{R}, \leq)$ are closed half-lines of the form $(r] := \{x \in \boldsymbol{R} : x \leq r\}$, where r ranges over real numbers. Any other ideal is an open half-line i.e., it is of the form $(r) := \{x \in \boldsymbol{R} : x < r\}$, $r \in \boldsymbol{R}$. After the identification of each real number r with the compact ideal $(r]$, $\boldsymbol{R}$ becomes the set of compact elements of $(Id(\boldsymbol{R}), \subseteq, \mathbf{1})$. The element (r) is smaller than $(r]$, for each r. But (r) is infinitesimaly close to the real $(r]$ $(= r)$ in the sense that (r) is greater than any real smaller than r. Thus the real number r is the unique cover of the infinitesimally closed element (r), for all $r \in \boldsymbol{R}$. The mapping which assigns to each real r the infinitesimal element (r) is of course an isomorphism between $(\boldsymbol{R}, \leq)$ and the family $\{(r) : r \in \boldsymbol{R}\}$ ordered by inclusion. □

Let $(P, \leq, \mathbf{1})$ be an integral poset. Let $(Q, \leq)$ be a directed-complete poset. Furthermore, let $F_0 : K(P) \to Q$ be a monotone function defined on the directed set of compact elements of $(P, \leq)$. Thus, for every pair $x, y \in K(P)$, $x \leq y$ implies that $F_0(x) \leq F_0(y)$. According to Theorem 4.6.10, there exists a unique order continuous function $F : P \to Q$, which is an extension of F_0. As $\mathbf{1} = \sup(K(P)$, we have that

$$F(\mathbf{1}) = \sup(\{F_0(x) : x \in K(P)\}).$$

The element $F(\mathbf{1})$ is called the *limit of F*.

The above observation gives rise to a certai plausible, general method of defining infinitistic concepts. This method can be briefly described as follows.

A general scheme of defining infinitistic concepts

Suppose we are given three components:

(a) a directed poset $(P_0, \leq)$,

(b) a directed complete poset $(Q, \leq)$,

(c) a monotone mapping $F_0 : P_0 \to Q$.

The elements of the set $F_0[P_0] := \{ F_0(a) : a \in P_0 \}$ are called *approximations to the defined concept*..

The component (a) determines

(d) the integral poset $(P, \leq, \mathbf{1})$ being the unique algebraic completion of $(P_0, \leq)$.

We adopt the following identification procedure, namely

(e) the poset $(P_0, \leq)$ is identified with the poset of compact elements of the completion $(P, \leq, \mathbf{1})$.

(a), (b) (c), (d) and (e) then determine

(f) the mapping $F : P \to Q$, the unique order continuous extension of F_0, where, for every $a \in P$:

$$F(a) := \sup(\{ F_0(x) : x \in K(a) \}).$$

Then

(g) the limit value $F(\mathbf{1}) = \sup(\{ F_0(x) : x \in K(P) \})$ defines the infinitistic object approximated by the values $F_0[P_0]$. □

The above scheme is thus rooted in the theory of order and does not primarily refer to primitive concepts having a distinct quantitative character like numbers or sets of numbers. Some interesting aplications of the above scheme can be obtained when one aditionaly assumes Martin's Axioms MA. This issue is not discussed here. In what follows MA is *not* assumed.

We shall present a number of examples showing particular, useful instances of the above scheme.

$(\wp(\boldsymbol{R}), \subseteq^d)$ stands for the directed-complete poset of subsets of the real line $\boldsymbol{R}$ with the *dual* inclusion $\subseteq^d := \supseteq$. Thus directed subsets in the sense of $(\wp(\boldsymbol{R}), \subseteq^d)$ are downward directed in the sense of $\subseteq$. Furthermore, the supremum of any directed subfamily of $(\wp(\boldsymbol{R}), \subseteq^d)$ is actually the set-theoretic *intersection* of this family. Any mapping with values belonging to the set $\wp(\boldsymbol{R})$ is called a *real-valued multi-function*. Customarily, any monotone mapping from a poset into $(\wp(\boldsymbol{R}), \subseteq^d)$ is called *anti-monotone*.

Let $(P, \leq)$ be a poset. By a *real multi-function* over $(P, \leq)$ we mean any mapping F from P to the powerset $\wp(\boldsymbol{R})$. Thus F assigns a *set* of real numbers to each element of P. A real multi-function $F : P \to \wp(\boldsymbol{R})$ is thus anti-monotone if, for every pair $a, b \in P$, $a \leq b$ implies $F(a) \supseteq F(b)$. Furthermore, if $(P, \leq)$ is directed-complete, a real multi-function $F : P \to \wp(\boldsymbol{R})$ is *continuous* (in the sense of $(\wp(\boldsymbol{R}), \subseteq^d)$) if it is anti-monotone and $F(\sup(D)) = \cap \{ F(d) : d \in D \}$ for every non-empty directed subset D of $(P, \leq)$. (If $F : P \to \wp(\boldsymbol{R})$ is anti-monotone and $D \subseteq$

P is directed, then the image $\{ F(d) : d \in D \}$ is a downward directed subfamily of $\wp(\boldsymbol{R})$.)

Let $(P, \leq, \mathbf{1})$ be an integral poset. Suppose that $F_0 : K(P) \to \wp(\boldsymbol{R})$ is an anti-monotone multi-function. Thus, for every pair $x, y \in K(P)$, $x \leq y$ implies that $F_0(x) \supseteq F_0(y)$. Let, furthermore, $F : P \to \wp(\boldsymbol{R})$ be the unique continuous extension of F_0 onto P. As $\mathbf{1} = \sup(K(P)$, we have that

$$F(\mathbf{1}) = \cap \{ F_0(x) : x \in K(P) \}.$$

W are mainly concerned with the case when $F(\mathbf{1})$ is a one-element set, i.e., $F(\mathbf{1}) = \{ r \}$ for some $r \in \boldsymbol{R}$. To give a better insight into the above notion, we shall examine more closely two examples : the standard definition of the limit of a sequence of real numbers and the definition of the Riemann integral of a continuous real function.

The limit of a sequence

By a *sequence* we mean any function f from ω to $\boldsymbol{R}$. $f \lceil n$ is the restriction of f to the set $\omega - n = \{ k \in \omega : k \geq n \}$. Let $P_0(f) := \{ f \lceil n : n \in \omega \}$. The elements of $P_0(f)$ are called *end segments* of f. Clearly, $P_0(f)$ is linearly ordered by the relation $\leq$, where for $u, v \in P_0(f)$:

$$u \leq v \text{ iff } u = f \lceil m, v = f \lceil n \text{ and } m \leq n.$$

It is clear that $(P_0(f), \leq)$ is isomorphic with $(\omega, \subseteq)$ and the sequence $\mathbf{0} := f$ is the least element in $(P_0(f), \leq)$.

Let $(P(f), \leq, \mathbf{1})$ be the algebraic completion of $(P_0(f), \leq)$. We identify $P_0(f)$ with the set of compact elements of $(P(f), \leq, \mathbf{1})$. The latter poset is thus isomorpic with the poset obtained from the former by adding the top element $\mathbf{1}$ to $P_0(f)$.

Given a segment $u = f \lceil n$, we define the closed set $F_0(u)$ of real numbers:

$$F_0(u) := \text{the closure of the image } f \lceil n [\omega - n]$$

$$(= \text{the closure of the set } \{ f(k) : k \in \omega \wedge k \geq n \}).$$

Evidently, $u \leq v$ implies $F_0(u) \supseteq F_0(v)$, for all $u, v \in P_0(f)$. Thus F_0 is an anti-monotone mapping from $P_0(f)$ to the set of closed subsets of $\boldsymbol{R}$. As $(P_0(f), \leq)$ is linear, and hence directed, the family $\{ F_0(u) : u \in P_0(f) \}$ has the finite intersection property, i.e., $F_0(u_1) \cap ... \cap F_0(u_n) \neq \varnothing$ for every non-empty, finite subfamily $\{ u_1, ..., u_n \}$ of $P_0(f)$.

According to Theorem 4.6.9, the mapping $F : P(f) \to \wp(\boldsymbol{R})$ given by

$$F(a) := \cap \{ F_0(u) : u \in P_0(f) \wedge u \leq a \}, \quad a \in P(f),$$

is the unique continuous extension of F_0 onto $(P(f), \leq, \mathbf{1})$. We are mainly concerned the case when the set $F(\mathbf{1}) := \cap\{F_0(u) : u \in P_0(f)\}$, the *limit set* of F_0, is a singleton. This set may be empty. E.g., this holds for the identity sequence f given by $f(n) := n$, for all $n \in \omega$. On the other hand, if f is an enumeration of the set of rational numbers, then $F(\mathbf{1}) = \boldsymbol{R}$. But if f is bounded, each of the sets $F_0(u)$, $u \in P_0(f)$, is compact. In this situation, the limit set $F(\mathbf{1}) = \cap\{F_0(u) : u \in P_0(f)\}$ is closed and non-empty. It is easy to see that the elements of $F(\mathbf{1})$ are the limits of all convergent subsequences of f.

Given a compact set $C \subseteq \boldsymbol{R}$, define

$$size(C) := \sup(\{ |x - y| : x, y \in C\}.$$

The following conditions are readily equivalent:

(1) *The sequence f has the limit* $\lim f$ (*in the usual sense).*

(2) *The limit set* $F(\mathbf{1})$ *is a singleton and* $F(\mathbf{1}) = \{\lim f\}$.

(3) *The sequence* $\{size(F_0(f\lceil n)\}_{n \in \omega}$ *is convergent to* 0. □

The Riemann integral

Let $I = [a, b]$ be a closed interval in the set $\boldsymbol{R}$ of real numbers, where $a, b \in \boldsymbol{R}$ and $a < b$. By a *finite partition* P of I we mean any non-empty finite family of closed intervals,

$$P = \{[a_i, b_i] : i = 0,..., n\},$$

such that $a_0 = a$, $b_n = b$, and $b_i = a_{i+1}$ for $i = 0,..., n-1$.

If P and Q are finite partitions of I, we define

> $P \leq Q$ iff Q is finer than P, that is, for every interval $B \in Q$ there exists an interval $A \in P$ such that $B \subseteq A$.

A finite partition P_2 is called an *intermediate successor* of a partition P_1 if P_2 is the effect of subdividing of only one interval $A \in P_1$ into two intervals. The remaining intervals belonging to P_1 are not divided and they also belong to P_2. Formally, if $P_1 = \{[a_i, b_i] : i = 0,..., n\}$, then there exist an integer k ($0 \leq k \leq n$) and a number c, where $a_k < c < b_k$, such that

$$P_2 = \{[a_i, b_i] : i < k\} \cup \{[a_k, c], [c, b_k]\} \cup \{[a_i, b_i] : k < i \leq n\}.$$

It is clear that for any finite partitions P and Q there holds: $P \leq Q$ iff either $P = Q$ or there exists a finite sequence of partitions $P_1,..., P_m$ such that $P_1 = P$, $P_m = Q$ and P_{i+1} is an intermediate successor of P_i for $i = 1,..., m-1$.

If $P = \{ [a_i, b_i] : i = 0,..., n \}$ is a finite partition of I, then $||P||$ is the length of the "largest" interval belonging to P. Formally, $||P|| := \sup(\{ b_i - a_i : i = 0,..., n \})$. It is clear that if $P \leq Q$, then $||Q|| \leq ||P||$. But, more importantly, for every partition P there exists a partition Q such that $||Q|| < ||P||$. To this end it suffices to subdivide *every* interval belonging to P into two proper intervals and define Q to be the resulting finite partition.

Let $\Pi_0(I)$ be the family of all finite partitions of the interval I.

Lemma 4.6.13. *The relation* $\leq$ *is a partial order on* $\Pi_0(I)$ and *the partition* **0** $= \{ I \}$ *consisting of the interval I only is the least element of the poset* $(\Pi_0(I), \leq)$. *Furthermore, the poset* $(\Pi_0(I), \leq)$ *is directed*.

The easy prof is omitted. □

Let $f : I \to \boldsymbol{R}$ be a real function defined on the interval I. For every finite partition $P \in \Pi_0(I)$, where $P = \{ [a_i, b_i] : i = 0,..., n \}$, we define the following set of real numbers called *Riemann sums of f corresponding to the partition P* :

$$F_0^f(P) := \{ \sum_{i=0}^{n} (b_i - a_i) \cdot f(\xi_i) : a_i \leq \xi_i \leq b_i, i = 0,..., n \}.$$

F_0^f is thus a well-defined mapping from $\Pi_0(I)$ into the set of all non-empty subsets of $\boldsymbol{R}$.

Lemma 4.6.14. *For every continuous function* $f : I \to \boldsymbol{R}$, *the mapping* $F_0^f : \Pi_0(I) \to \wp(\boldsymbol{R})$ *is anti-monotone*.

Proof. It suffices to show that for any partitions P_1 and P_2, if P_2 is an intermediate successor of P_1, then $F_0^f(P_2) \subseteq F_0^f(P_1)$. But the last inclusion imediately follows from the following observation :

Claim 1. *Let* $f : I \to \boldsymbol{R}$ *be a continuous function defined on a closed interval* $I = [a, b]$. *Let c be a number such that* $a < c < b$ *and let* ξ_1 *and* ξ_2 *be numbers such that* $a \leq \xi_1 \leq c$ *and* $c \leq \xi_2 \leq b$. *Then there exists a number* ξ, $a \leq \xi \leq b$, *such that*

$$(b - a) \cdot f(\xi) = (c - a) \cdot f(\xi_1) + (b - c) \cdot f(\xi_2).$$

Proof of the claim. We consider two cases.

Case 1. $f(\xi_1) \leq f(\xi_2)$.

Then $(c - a) f(\xi_1) \leq (c - a) f(\xi_2)$ and, consequently,

$$(c - a) f(\xi_1) + (b - c) f(\xi_2) \leq (c - a) f(\xi_2) + (b - c) f(\xi_2) = (b - a) f(\xi_2).$$

Analogously,

$$(b - a) f(\xi_1) = (c - a) f(\xi_1) + (b - c) f(\xi_1) \le (c - a) f(\xi_1) + (b - c) f(\xi_2).$$

It follows from the above that

(*) $$(b - a) f(\xi_1) \le (c - a) f(\xi_1) + (b - c) f(\xi_2) \le (b - a) f(\xi_2).$$

Since the function $(b - a) f(x)$ is continuous on I, it takes any intermediate value between the numbers $(b - a) f(\xi_1)$ and $(b - a) f(\xi_2)$. This, in the presence of (*), implies that there exists a number ξ, $a \le \xi \le b$, such that

$$(b - a) \cdot f(\xi) = (c - a) \cdot f(\xi_1) + (b - c) \cdot f(\xi_2).$$

Case 2. $f(\xi_2) \le f(\xi_1)$.

In this situation the proof is similar. (We omit it.)

The above two cases yield the thesis and conclude the proof of the lemma. □

Note. The above proof shows that the thesis of the claim (and at the same time – the thesis of the lemma) is reached if one only assumes that f has the Darboux property. □

Given subsets A, $B \subseteq \boldsymbol{R}$, define :

$$A \oplus B := \{ a + b : a \in A \wedge b \in B \}.$$

The operation $\oplus$ is commutative and associative. The definition of $\oplus$ is inductively extended (in the obvious way) onto finite non-empty families of subsets of $\boldsymbol{R}$. It is easy to see that if A and B are closed sets and one of them is compact, then $A \oplus B$ is closed as well. We note that for any closed intervals $[a, b]$ and $[c, d]$:

$$[a, b] \oplus [c, d] \subseteq [a + c, b + d].$$

Lemma 4.6.15. *Let* $P = \{ [a_i, b_i] : i = 0,..., n \}$ *be a finite partition of a closed interval* I. *Given a function* $f : I \to \boldsymbol{R}$, *define*:

$$m_i := \inf(\{ f(\xi) : a_i \le \xi \le b_i \}), \qquad M_i := \sup(\{ f(\xi) : a_i \le \xi \le b_i \}).$$

If the function f is continuous, *then*

$$F_0^f(P) = [\lambda_0 m_0, \lambda_0 M_0] \oplus \ldots \oplus [\lambda_n m_n, \lambda_n M_n],$$

where $\lambda_i := b_i - a_i$ *for* $i = 0,..., n$. *Consequently*, , $F_0^f(P)$ *is a closed and bounded non-empty subset of* $\boldsymbol{R}$.

Proof. The above equality readily follows from the fact that f has the Darboux property. As $[\lambda_0 m_0, \lambda_0 M_0] \oplus \ldots \oplus [\lambda_n m_n, \lambda_n M_n]$ is closed and

$$[\lambda_0 m_0, \lambda_0 M_0] \oplus \ldots \oplus [\lambda_n m_n, \lambda_n M_n] \subseteq$$

$$[\lambda_0 m_0 + \ldots + \lambda_n m_n, \lambda_0 M_0 + \ldots + \lambda_n M_n],$$

the second statement follows. □.

Given a closed interval $I = [a, b]$, we define the directed poset $(\Pi_0(I), \leq)$ of finite partitions of I. Let $(\Pi(I), \leq, \mathbf{1})$ be the unique integral poset being the algebraic completion of $(\Pi_0(I), \leq)$. Hence, without loss of generality, we may assume that $(\Pi_0(I), \leq)$ itself coincides with the poset of compact elements of $(\Pi(I), \leq, \mathbf{1})$. Furthermore, given a continuous function $f : I \to \boldsymbol{R}$, define, as above, the anti-monotone mapping $F_0^f : \Pi_0(I) \to \wp(\boldsymbol{R})$. Let $F^f : \Pi(I) \to \wp(\boldsymbol{R})$ be the unique continuous extension of F_0^f onto $\Pi(I)$. (In virtue of Theorem 4.6.9, such an extension exists.) Thus

$$F^f(a) = \cap \{ F_0^f(P) : P \in \Pi_0(I) \text{ and } P \leq a \},$$

for all $a \in \Pi(I)$. In particular, the limit set

$$F^f(\mathbf{1}) := \cap \{ F_0^f(P) : P \in \Pi_0(I) \}$$

is well-defined.

Theorem 4.6.16. *The set $F^f(\mathbf{1})$ is a singleton. More exactly,*

$$F^f(\mathbf{1}) = \{ \int_a^b f(x)\, dx \}.$$

Proof. The integral on the right side of the above equality is defined in the standard way. Thus a number α is the definite integral of a function f over $I = [a, b]$, in symbols

$$\alpha = \int_a^b f(x)\, dx \},$$

if for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every partition $P = \{ [a_i, b_i] : i = 0,\ldots, n \} \in \Pi_0(I)$ for which $||P|| < \delta$ and for any numbers ξ_i, where $a \leq \xi_i \leq b_i$, for $i = 0, \ldots, n$, there holds

$$\left| \left(\sum_{i=0}^{n} (b_i - a_i) \cdot f(\xi_i) \right) - \alpha \right| < \varepsilon$$

(See any textbook on calculus.)

Since the mapping $F_0^f : \Pi_0(I) \to \wp(\boldsymbol{R})$ is anti-monotone and $(\Pi_0(I), \leq)$ is directed, it follows that the family of sets

$$\{ F_0^f(P) : P \in \Pi_0(I) \}$$

has the finite intersection property, i.e., $F_0^f(P_1) \cap ... \cap F_0^f(P_n) \neq \emptyset$ for every non-empty, finite subfamily $\{P_1,..., P_n\}$ of $\Pi_0(I)$. Furthermore, each set $F_0^f(P)$, $P \in \Pi_0(I)$, is a closed subset of the closed set $F_0^f(\mathbf{0})$, where $\mathbf{0} = \{I\}$ is the least partition in $\Pi_0(I)$. $F_0^f(\mathbf{0})$ is the interval $[(b-a)\cdot m, [(b-a)\cdot M]$, where

$$m := \inf(\{ f(\xi) : a \le \xi \le b\}) \quad \text{and} \quad M := \sup(\{ f(\xi) : a \le \xi \le b\}).$$

$F_0^f(\mathbf{0})$ is therefore a compact subset of $\boldsymbol{R}$. But in any compact space the intersection of any non-empty family of closed subsets with the finite intersection property is non-empty. Consequently, the intersection of the family $\{ F_0^f(P) : P \in \Pi_0(I)\}$ is non-empty. But, in fact, the set $\cap \{ F_0^f(P) : P \in \Pi_0(I)\}$ is a singleton. This, roughly, follows from the fact that if partitions P grow large in the poset $(\Pi_0(I), \le)$, then the closed sets $F_0^f(P)$ get smaller and smaller in size. The proof runs here as follows. Fix a postive real number $\varepsilon > 0$. As f is uniformly continuous on I, there exists a positive $\delta > 0$ such that, for all $x, y \in I$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Let $P = \{ [a_i, b_i] : i = 0,..., n\}$ be a finite partition of the interval I such that $||P|| = \sup(\{ b_i - a_i : i = 0,..., n\}) < \delta$, i.e., the lengthts of the intervals belonging to P are less than δ. Let

$$m_i := \inf(\{ f(\xi) : a_i \le \xi \le b_i\}) \text{ and } M_i := \sup(\{ f(\xi) : a_i \le \xi \le b_i\})$$

for: $i = 0,..., n$. Putting $\lambda_i := b_i - a_i$ *for* $i = 0,..., n$, we have, by Lemma 4.6.15, that

$$F_0^f(P) = [\lambda_0 m_0, \lambda_0 M_0] \oplus ... \oplus [\lambda_n m_n, \lambda_n M_n]$$

$$\subseteq [\lambda_0 m_0 + ... + \lambda_n m_n, \lambda_0 M_0 + ... + \lambda_n M_n].$$

The length of the last interval is bounded from above by $\lambda_0 \varepsilon + ... + \lambda_n \varepsilon$. But

$$\lambda_0 \varepsilon + ... + \lambda_n \varepsilon = \varepsilon(\lambda_n + ... + \lambda_0) =$$

$$\varepsilon((b_n - a_n) + (b_{n-1} - a_{n-1}) + ... + (b_0 - a_0)) =$$

$$\varepsilon(b_n - a_0) = \varepsilon(b - a).$$

Thus taking an arbitrary ε, we always find a partition P such that the size of $F_0^f(P)$ (i.e., the distances between the points belonging to $F_0^f(P)$) is bounded by $\varepsilon(b - a)$. This proves that the set $\cap \{ F_0^f(P) : P \in \Pi_0(I)\}$ contains only one point.

From the above remarks the theorem follows. □

4.7. The Jordan measure.

Given two closed intervals $[a, b]$ and $[c, d]$, where $a, b, c, d \in \boldsymbol{R}$, we define the closed rectangle $\Delta := [a, b] \times [c, d]$.

For every finite partition

$$P = \{ [a_i, a_{i+1}] : i = 0,..., m \}$$

of the interval $[a, b]$ and for every finite partition

$$Q = \{ [c_j, c_{j+1}] : j = 0,..., n \}$$

of the interval $[c, d]$, we define

(1) $$P \times Q := \{ [a_i, a_{i+1}] \times [c_j, c_{j+1}] : 0 \leq i \leq m, 0 \leq j \leq n \}.$$

$P \times Q$ is a partition of the rectangle Δ.

Given two finite partitions P_1 and P_2 of $[a, b]$ and partitions Q_1 and Q_2 of $[c, d]$, we write:

$$P_1 \times Q_1 \leq P_2 \times Q_2 \text{ iff } P_1 \leq P_2 \text{ and } Q_1 \leq Q_2,$$

i.e., $P_2 \times Q_2$ is finer than $P_1 \times Q_1$ iff P_2 is finer than P_1 and Q_2 is finer than Q_1.

As in the one dimensional case, we say that a partition $P_2 \times Q_2$ is an *intermediate successor* of a partition $P_1 \times Q_1$ if P_2 is the effect of subdividing of only one rectangle $A \in P_1 \times Q_1$ into two rectangles. The remaining rectangles belonging to $P_1 \times Q_1$ are not divided and they also belong to $P_2 \times Q_2$. Formally, if

$$P_1 \times Q_1 = \{ [a_i, a_{i+1}] \times [c_j, c_{j+1}] : 0 \leq i \leq m, 0 \leq j \leq n \},$$

then

either there exist an integer k ($0 \leq k \leq m$) and a number c, where $a_k < c < a_{k+1}$, such that

$$P_2 = \{ [a_i, a_{i+1}] : i < k \} \cup \{ [a_k, c], [c, a_{k+1}] \} \cup \{ [a_i, a_{i+1}] : k < i \leq m \}$$

and $Q_2 = Q_1$

or there exist an integer l ($0 \leq l \leq n$) and a number e, where $c_l < e < c_{l+1}$, such that

$$Q_2 = \{ [c_j, c_{j+1}] : j < l \} \cup \{ [c_l, e], [e, c_{l+1}] \} \cup \{ [c_j, c_{j+1}] : l < j \leq n \}$$

and $P_2 = P_1$.

Thus $P_2 \times Q_2$ is an intermediate successor of a partition $P_1 \times Q_1$ if either P_2 is an intermediate successor of P_1 (as partitions of $[a, b]$) and $Q_2 = Q_1$ or Q_2 is an intermediate successor of Q_1 and $P_2 = P_1$.

It is clear that for any finite partitions $P_1 \times Q_1$ and $P_2 \times Q_2$ of Δ there holds: $P_1 \times Q_1 \leq P_2 \times Q_2$ iff either $P = Q$ or there exists a finite sequence of partitions $P^1 \times Q^1, \ldots, P^m \times Q^m$ of Δ such that $P^1 \times Q^1 = P_1 \times Q_1$, $P^m \times Q^m = P_2 \times Q_2$ and the partition $P^{i+1} \times Q^{i+1}$ is an intermediate successor of $P^i \times Q^i$ for $i = 1, \ldots, m-1$.

It easily follows from the above facts that $\leq$ is a partial order relation on the set $\Pi_0(\Delta)$ of finite partitions (1) of the rectangle Δ. Furthermore the poset $(\Pi_0(\Delta), \leq)$ is directed and the trivial partition $\{\Delta\}$, denoted also by **0**, is the least element in $(\Pi_0(\Delta), \leq)$.

Let Z be a subset of the plane $\mathbf{R}^2$ included in the closed rectangle $\Delta = [a, b] \times [c, d]$. Given a partition (1) of Δ we define:

(2) $$F_0^Z(P \times Q) := [\underline{u}, \overline{u}],$$

where $\underline{u}$, $\overline{u}$ are real numbers defined as follows:

$$\underline{u} := \sum\{ (a_{i+1} - a_i)(c_{j+1} - c_j): [a_i, a_{i+1}] \times [c_j, c_{j+1}] \subseteq Z, 0 \leq i \leq m, 0 \leq j \leq n\},$$

$$\overline{u} := \sum\{ (a_{i+1} - a_i)(c_{j+1} - c_j): Z \cap ([a_i, a_{i+1}] \times [c_j, c_{j+1}]) \neq \emptyset, 0 \leq i \leq m, 0 \leq j \leq n\}.$$

It is clear that $\underline{u} \leq \overline{u}$.

Proposition 4.7.1. *For every $Z \subseteq \Delta$, F_0^Z is a monotone mapping from the directed poset $(\Pi_0(\Delta), \leq)$ into the directed-complete poset $(CII(\mathbf{R}), \subseteq^d)$ of closed bounded intervals in $\mathbf{R}$, ordered by the dual inclusion.*

The proof is immediate. It suffices to show that if $P_2 \times Q_2$ is an intermediate successor of $P_1 \times Q_1$, then $F_0^Z(P_1 \times Q_1) \supseteq F_0^Z(P_2 \times Q_2)$. To this end we conider wo cases.

Case 1. P_2 is an intermediate successor of P_1 and $Q_2 = Q_1$.

Let $P_1 = \{ [a_i, a_{i+1}] : i = 0, \ldots, m \}$. There exists k ($0 \leq k \leq m$) and a number a, where $a_k < a < a_{k+1}$, such that

$$P_2 = \{ [a_i, a_{i+1}] : i < k \} \cup \{ [a_k, a], [a, a_{k+1}] \} \cup \{ [a_i, a_{i+1}] : k < i \leq m \}$$

and $Q_2 = Q_1$, where $Q_1 = \{ [c_j, c_{j+1}] : j = 0, \ldots, n \}$.

Put:

$$F_0^Z(P_1 \times Q_1) = [\underline{u}_1, \overline{u}_1], \quad F_0^Z(P_2 \times Q_2) = [\underline{u}_2, \overline{u}_2].$$

Furthermore, let $\Delta_{ij} := [a_i, a_{i+1}] \times [c_j, c_{j+1}]$ for $i = 0, \ldots, m$, $j = 0, \ldots, n$. Thus

$$\underline{u}_1 = \sum\{ (a_{i+1} - a_i)(c_{j+1} - c_j): \Delta_{ij} \subseteq Z, 0 \leq i \leq m, 0 \leq j \leq n\};$$

$\overline{u}_1 = \sum\{ (a_{i+1} - a_i)(c_{j+1} - c_j): Z \cap \Delta_{ij} \neq \emptyset, 0 \leq i \leq m, 0 \leq j \leq n\};$

$$\begin{aligned}\underline{u}_2 = \; & \sum\{ (a_{i+1} - a_i)(c_{j+1} - c_j): \Delta_{ij} \subseteq Z, 0 \leq i < k, 0 \leq j \leq n\} + \\ & \sum\{ (a - a_k)(c_{j+1} - c_j): \Delta^L_{ka} \subseteq Z, 0 \leq j \leq n\} + \\ & \sum\{ (a_{k+1} - a)(c_{j+1} - c_j): \Delta^R_{ka} \subseteq Z, 0 \leq j \leq n\} + \\ & \sum\{ (a_{i+1} - a_i)(c_{j+1} - c_j): \Delta_{ij} \subseteq Z, k < i \leq m, 0 \leq j \leq n\},\end{aligned}$$

where $\Delta^L_{ka} := [a_k, a] \times [c_j, c_{j+1}]$, $\Delta^R_{ka} := [a, a_{k+1}] \times [c_j, c_{j+1}]$;

$$\begin{aligned}\overline{u}_2 = \; & \sum\{ (a_{i+1} - a_i)(c_{j+1} - c_j): Z \cap \Delta_{ij} \neq \emptyset, 0 \leq i < k, 0 \leq j \leq n\} + \\ & \sum\{ (a - a_k)(c_{j+1} - c_j): Z \cap \Delta^L_{ka} \neq \emptyset, 0 \leq j \leq n\} + \\ & \sum\{ (a_{k+1} - a)(c_{j+1} - c_j): Z \cap \Delta^R_{ka} \neq \emptyset, 0 \leq j \leq n\} + \\ & \sum\{ (a_{i+1} - a_i)(c_{j+1} - c_j): Z \cap \Delta_{ij} \neq \emptyset, k < i \leq m, 0 \leq j \leq n\}.\end{aligned}$$

It is clear that for each j, $0 \leq j \leq n$,

$$(a - a_k)(c_{j+1} - c_j) + (a_{k+1} - a)(c_{j+1} - c_j) = (a_{k+1} - a_k)(c_{j+1} - c_j).$$

As $\Delta_{kj} = \Delta^L_{ka} \cup \Delta^R_{ka}$, we have that

$$\Delta_{kj} \subseteq Z \text{ iff } \Delta^L_{ka} \subseteq Z \text{ and } \Delta^R_{ka} \subseteq Z.$$

Consequently, $\underline{u}_1 = \underline{u}_2$.

As

$$Z \cap \Delta_{kj} \neq \emptyset \text{ iff } Z \cap \Delta^L_{ka} \neq \emptyset \text{ or } Z \cap \Delta^R_{ka} \neq \emptyset,$$

It follows that

$$\begin{aligned}& \sum\{ (a - a_k)(c_{j+1} - c_j): Z \cap \Delta^L_{ka} \neq \emptyset, 0 \leq j \leq n\} + \\ & \sum\{ (a_{k+1} - a)(c_{j+1} - c_j): Z \cap \Delta^R_{ka} \neq \emptyset, 0 \leq j \leq n\} \\ \leq \; & \sum\{ (a_{k+1} - a_k)(c_{j+1} - c_j): \Delta_{kj} \subseteq Z, 0 \leq j \leq n\}.\end{aligned}$$

Consequently, $\overline{u}_2 \leq \overline{u}_1$. Thus

$$F_0^Z(P_1 \times Q_1) = [\underline{u}_1, \overline{u}_1] \supseteq [\underline{u}_1, \overline{u}_2] = [\underline{u}_2, \overline{u}_2] = F_0^Z(P_2 \times Q_2).$$

Case 2. Q_2 is an intermediate successor of Q_1 and $P_2 = P_1$.

This case is handled similarly to Case 1.

In both the cases we obtain the inclusion $F_0^Z(P_1 \times Q_1) \supseteq F_0^Z(P_2 \times Q_2)$. □

Let ($\Pi(\Delta)$, ≤, **1**) be the unique integral completion of the directed poset ($\Pi_0(\Delta)$, ≤). We identify $\Pi_0(\Delta)$ with the set of compact elements of ($\Pi(\Delta)$, ≤, **1**). Furthermore, given a set $Z \subseteq \Delta$, let F^Z be the unique order continuous extension of F_0^Z onto ($\Pi(\Delta)$, ≤, **1**). Hence $F^Z(\mathbf{1}) = \cap \{ F_0^Z(P \times Q) : P \times Q \in \Pi_0(\Delta)\}$.

Theorem 4.7.2. *Let Z be a subset of* Δ. *Z is Jordan measurable iff* $F^Z(\mathbf{1})$ *is a singleton. Furthermore*, *if* $F^Z(\mathbf{1})$ *is a singleton, then its only element is the Jordan measure of Z.*

Proof. The following convention is adopted in this section. If x_t, $t \in T$, is a finite sequence of real numbers (repetitions are allowed), then $\sum\{ x_t : t \in T\}$ denotes the sum of this sequence.

We recall that the *outer Jordan measure* $m_e(Z)$ of Z is defined to be the infimum of the numbers $\overline{u}$, where $\overline{u}$ is defined for each partition $P \times Q$ as in (2), i.e.,

$$\overline{u} := \sum\{ (a_{i+1} - a_i)(c_{j+1} - c_j): Z \cap ([a_i, a_{i+1}] \times [c_j, c_{j+1}]) \neq \emptyset, 0 \le i \le m, 0 \le j \le n\}.$$

The *inner Jordan measure* $m_i(Z)$ of Z is defined to be the supremum of the numbers $\underline{u}$, where

$$\underline{u} := \sum\{ (a_{i+1} - a_i)(c_{j+1} - c_j): [a_i, a_{i+1}] \times [c_j, c_{j+1}] \subseteq Z, 0 \le i \le m, 0 \le j \le n\}.$$

Z is *Jordan measurable* if $m_e(Z) = m_i(Z)$. In this case the number $m(Z) := m_e(Z)$ (= $m_i(Z)$) is called the *Jordan measure of Z*. It is a well-known fact that Jordan measurable subsets of any rectangle Δ form a Boolean algebra of sets. Every open subset of Δ is Jordan measurable.

We have sketched here the definition of the Jordan measure for the two-dimensional case, i.e., for subsets of $\boldsymbol{R}^2$. Furthermore, to simplify the matters, we consider only bounded subsets of $\boldsymbol{R}^2$. It is quite obvious that the above definitions transfer smoothly to an arbitrary finite dimension k, $k \ge 1$. We note that the set of rational numbers is *not* Jordan measurable.

(⇒). Assume Z is Jordan measurable. So $m_e(Z) = m_i(Z)$. We have: $F^Z(\mathbf{1}) = \cap \{ F_0^Z(P \times Q) : P \times Q \in \Pi_0(\Delta) \}$. But each set $F_0^Z(P \times Q)$ is a closed, bounded and non-empty interval. Hence $F^Z(\mathbf{1})$ itself is a non-empty, bounded and closed interval in $\boldsymbol{R}$. We show that $F^Z(\mathbf{1})$ is actually a trivial interval, that is, it reduces to a singleton. Suppose otherwise. So $F^Z(\mathbf{1}) = [x, y]$ where $x < y$. It follows from the definition of $F^Z(\mathbf{1})$ that for every partition $P \times Q$ there holds: $\underline{u} \le x < y \le \overline{u}$, where $\underline{u}$ and $\overline{u}$ are defined (for $P \times Q$) as above. As the poset ($\Pi_0(\Delta)$, ≤) is directed, it follows that for every $\varepsilon > 0$ there exists a partition $P \times Q \in \Pi_0(\Delta)$ such that $m_i(Z) - \varepsilon < \underline{u}$ and $\overline{u} < m_e(Z) + \varepsilon$. Hence, taking ε to be sufficiently small, we have

$$m_i(Z) < \underline{u} + \varepsilon < x + \varepsilon < y - \varepsilon \le \overline{u} - \varepsilon < m_e(Z),$$

which gives that $m_i(Z) < m_e(Z)$. So Z is not Jordan measurable. A contradiction. The obtained contradiction shows that $F^Z(\mathbf{1})$ is a singleton.

($\Leftarrow$). We assume that $F^Z(\mathbf{1})$ is a singleton. Suppose Z is not Jordan measurable. So $m_i(Z) < m_e(Z)$. It follows that for every partition $P \times Q \in \Pi_0(\Delta)$, we have that $\underline{u} \leq m_i(Z) < m_e(Z) \leq \overline{u}$, where $\underline{u}$ and $\overline{u}$ are defined as above for each $P \times Q$. Hence the proper interval $[m_i(Z), m_e(Z)]$ is included in $F_0^Z(P \times Q)$, for every partition $P \times Q \in \Pi_0(\Delta)$. As $F^Z(\mathbf{1}) = \cap \{ F_0^Z(P \times Q) : P \times Q \in \Pi_0(\Delta)\}$, it follows that $F^Z(\mathbf{1})$ includes the proper interval $[m_i(Z), m_e(Z)]$. So $F^Z(\mathbf{1})$ is not a singleton.

The second statement of the theorem is obvious. □

$|X|$ stands for the cardinality of a set X.

Given a finite partition $P \times Q \in \Pi_0(\Delta)$, where $\Delta = [a, b] \times [c, d]$, $P = \{ [a_i, a_{i+1}] : i = 0,..., m \}$, $Q = \{ [c_j, c_{j+1}] : j = 0,..., n \}$, and a set $Z \subseteq \Delta$, we define the fractions:

$$f^+_Z(P \times Q) := |\{(i, j) : Z \cap \Delta_{ij} \neq \emptyset, 0 \leq i \leq m, 0 \leq j \leq n \}| \,/\, mn,$$

$$f^-_Z(P \times Q) := |\{(i, j) : \Delta_{ij} \subseteq Z, 0 \leq i \leq m, 0 \leq j \leq n \}| \,/\, mn.$$

Note that mn is the total number of the rectangles Δ_{ij}. The functions f^+_Z and f^-_Z are monotone on the poset $(\Pi_0(\Delta), \leq)$. Both f^+_Z and f^-_Z thus have unique extensions F^+_Z and F^-_Z, respectively, being order continuous mappings from the integral poset $(\Pi(\Delta), \leq, \mathbf{1})$ to the directed-complete poset $([0,1], \leq)$, where $[0,1]$ stands for the unit closed interval.

Theorem 4.7.2. *A set $Z \subseteq \mathbf{R}^2$ is Jordan measurable iff $F^+_Z(\mathbf{1}) = F^-_Z(\mathbf{1})$. Furthermore, if Z is Jordan measurable, then $m(Z) = (b - a)\cdot(d - c)\cdot F^+_Z(\mathbf{1}) = (b - a)\cdot(d - c)\cdot F^-_Z(\mathbf{1})$.* □

The number $F^+_Z(\mathbf{1})$ is interpreted as the probability that a randomly selected subrectangle of Δ is *not* disjoint with Z.

$F^-_Z(\mathbf{1})$ is the probability that that a randomly selected subrectangle of Δ is contained in Z.

Thus $Z \subseteq \mathbf{R}^2$ is Jordan measurable iff the two probabilities are equal.

It is relatively easy to provide an array of results similar to the previous ones, thus indicating the role of Theorems 4.6.3 and 4.6.5 as a tool in defining various, inherently infinitistic notions. We do not investigate this issue thoroughly because it is beyond the scope of this book.