First draft

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LOGIC

a short guide to its
main concepts, and its philosophical foundations

About the guide

Logic in the form it acquired at the beginning of the twenty century is science on the formal structure of deductive theories, notably mathematical theories presented in the form of axiom systems. The key concept in terms of which such a structure is defined is that of logical consequence. In the most concise way, logic might be then defined as the theory of consequence relation.

A widely spread cliché that logic sets criteria of "logical thinking" reflects rather naïve idea that rigorous and precise thinking is matter of knowing formal principles of reasoning rather than having a deep enough grasp of its subject. But this is not so. We prize one for "logical thinking" if, in our opinion, one controls all subtleties of the discourse, one is aware of various significant but easy to be overlooked aspects of the discussed topic, one evaluates correctly the hierarchy and relevance of issues being examined, etc., etc. All these criteria of "logical thinking" concern the matter of reasoning rather than its form.

And yet, in a way, logic concerns "logical thinking" in the naïve sense of the word for it provides guidance and tools needed for grasping the formal structure of any rational discourse whatsoever. Needless to say thus, as long as the formal structure of a discourse remains unclear, the question of which of the propositions of the analyzed discourse follow from which may not have a conclusive answer. One of the key issues we are going to discuss in this text is how and under which conditions the principles and techniques of formalized (mathematical) discourse can be applied in informal reasoning. Thus we are going to depart from the idea that the chief domain of application of logic is mathematics and examine relevance of logical concepts and techniques for analysis of a discourse of any kind whatsoever.

The ideas presented and discussed in this guide were presented in an extended form in the textbook *Lectures on logic with elements of the theory of knowledge* (Scholar, Warsaw, 2003) at present available only in Polish (its original title is *Wykłady z logiki z elementami teorii wiedzy*). Its English version is being prepared.

1. The concept of set

1.1. The Common-Sense vs. Formal Idea of a Set

The common-sense idea of set might seem to be simple and unproblematic. In fact, it is not. At the same time, unless we do not mind committing the fallacy known as *ignotum per ignotum* (defining "unknown by unknown"), no definition of a set is available. For instance, to explain that "a set is a collection of objects" is to explain nothing. The idea of a collection is not a bit clearer than that of a set.

The right way to handle the problem is to explicate what is a set by suitably selected meaning postulates (axioms). We are not going to present set theory as an axiom system, however. Instead, we shall adopt the following approach. We shall clarify the intuitive meaning of the term "set" with the help of selected "principles" meant to describe the most characteristic properties of sets. We prefer to use the term "principle" rather than "axiom" for in a specific axiom system they might be ***theorems*** (i.e. logical consequences of axiom) rather than axioms.

1.2. The Extensionality Principle

Consider the following question. Let S be the set of all Warsaw streets and let D be the set of all Warsaw districts. Is S = D? Warsaw can be viewed both as a "unit" combined of different streets a "unit" combined of different districts. This suggests that S = D. And yet S ≠ D, in virtue of the following commonly accepted principle (*iff* abbreviates *if and only if*):

> **1.2.1. EXTENSIONALITY PRINCIPLE (INFORMAL VERSION)**: *Sets A, B are the same iff they consist of the same elements.*

Indeed, the elements of S are streets, and those of D are districts. No street is a district and no district is a street. So S and D consist of different elements and thus they are different.

Of course, Warsaw is a "unit" formed by its various parts. The term "unit" that appears in this statement should not be interpreted as "set," however. Rather it should be interpreted as a ***system***, meant to be a set whose elements satisfy certain either implicitly accepted or explicitly stated conditions.

1.3. The Formalized Version of Extensionality Principle

We have stated Extensionality Principle in a rather loose language. Mathematicians apply informal language fairly often but always with considerable cautious. The more sophisticated is a mathematical proposition the more likely is that its loose version might not be clear enough.

A rigorous version of Extensionality Principle is the following ($x \in A$ is the commonly accepted notation for x is an element of A):

> **1.3.1. EXTENSIONALITY PRINCIPLE (RIGOROUS VERSION)**: *Let A and B be sets. Then A = B iff for every object x, x ∈ A iff x ∈ B.*

Replace *iff* by ≡ and replace *for every* by ∀. Assume that variables *A, B, C,…* represent sets while variables *x, y, z, …* represent any objects whatsoever. Now, the Extensionality Principle might be stated as follows:

> **1.3.2. EXTENSIONALITY PRINCIPLE (FORMALIZED VERSION)**:
> $A = B \equiv \forall x\, (x \in A \equiv x \in B)$

Some more notations will be useful. In order to state that x is a set, we shall write **`set`**(x). Incidentally, this notation sets a general pattern we shall apply on numerous occasions. Thus e.g. **`mathematician`**(x) will serve as formalized counterpart of *x is a mathematician*, **`brave-soldier`**(x) will serve as formalized notation for *x is a brave soldier*, etc.

1.4. Logical Notation

Rather than introducing more logical notation when this is needed, let us list all logical symbols we are going to use. They are the following (expressions in parentheses indicate the intended meaning of the symbols):

> **1.4.1. LIST OF LOGICAL TERMS AND THEIR SYMBOLS**:
> **Connectives**: ***negation*** ¬ (*not*), ***implication*** → (*if...then*), ***conjunction*** ∧ (*and*), ***disjunction*** ∨ (*or*), ***equivalence*** ≡ (*if and only if*)
> **Quantifiers** (x stands for an arbitrary variable)**:** ***universal*** ∀*x* (*for all x*), ***existential*** ∃*x* (*there is x such that*)
> **Identity predicate** = (*is the same as*)

Logical terms (both in their symbolic and verbal form) are often referred to as logical "constants." The terms listed above are known as ***classical logical constants***.

Keep in mind that the term "classical logical constant" applies to the terms listed in 1.4.1 only if the terms are understood in the way accepted in "classical logic" (see XXX).

1.5. Two methods of defining sets and some notation

Listing method: One way to define a set is to list the objects of which the set is composed. By $\{a_1,a_2,...,a_n\}$ we shall denote the set whose elements are $a_1,a_2,...,a_n$.

Rule method: Various sets can be defined to be the set of those objects that satisfy a specific condition $\Phi(x)$ (e.g. are mathematicians if $\Phi(x)$ is **`mathematician`**(x)). The set thus defined will be denoted by $\{x|\Phi(x)\}$. Surprisingly enough (see the next Section), for some conditions of the form $\Phi(x)$ the objects that satisfy that conditions do not form a set (an assumption that they do yields contradiction).

The set $\{a_1,a_2,...,a_n\}$ is the set whose elements are $a_1,a_2,...,a_n$ and $\{x|\Phi(x)\}$ is the set of all objects x that satisfy condition $\Phi(x)$. Thus as long as we do not mind using informal language, the notation we have introduced is redundant.

Along with notation $\{a_1,a_2,...,a_n\}$ on various occasions we shall use another one, namely $(a_1,a_2,...,a_n)$. The latter stands for the "sequence" formed by arranging objects a_1, $a_2,...,a_n$ in the indicated order; the first element of the sequence is a_1, the second is a_2, etc., the last a_n. Two elements of the sequence need not to be different; thus e.g. a_3 might be the same as a_5. Two sequences $(a_1,a_2,...,a_n)$ and $(b_1,b_2,...,b_m)$ are identical iff $n = m$ (i.e. they are of the same length) and fore every i, $a_i = b_i$. The term "sequence" will be applied interchangeably with ***n-tuple*** (n being the length of the sequence).

Clearly n-tuple $(a_1,a_2,...,a_n)$ is not the same as the set $\{a_1,a_2,...,a_n\}$. We trust the reader to explain why.

1.6. Unit Sets and Non-Reflexivity Principle

A ***unit set*** is a set of the form $\{a\}$. Is a unit set $\{a\}$ the same object as its element a? The common-sense idea of a set provides no guidance for dealing with this issue. Note that the question of whether $\{a\} = a$ may be treated as a special case of a more general one: is it possible that a set is an element of itself? Most of the set-theorist subscribe to the following

1.5.1. NON-REFLEXIVITY PRINCIPLE: *No set is an element of itself.*

Then the symbolic version of 1.5.1 is: $\forall x(\mathbf{set}(x) \rightarrow \neg(x{\in}x)$. Incidentally instead of $\neg(x{\in}y)$ one often writes $x{\notin}y$. Thus $\forall x(\mathbf{set}(x) \rightarrow x{\notin}x$ is an alternative way of formalizing 1.5.1.

One of immediate consequences of Non-Reflexivity Principle is that for no x, $x = \{x\}$; thus, for instance, $0 \neq \{0\}$, $\{0\} \neq \{\{0\}\}$,… New York ≠ {New York}, {New York} ≠ {{New York}}, etc. Indeed, if $x = \{x\}$ then $x \in \{x\}$, contrary to Non-Reflexivity Principle.

Another consequence of Non-Reflexivity is that there is no set of all sets. Indeed, Suppose that there is such a set. Then it is one of its elements, which by Non-reflexivity Principle is impossible. Note that since there is no set of all sets then condition $\mathbf{set}(x)$ does not define any set and hence the notation $\{x|\mathbf{set}(x)\}$ stands for no object.

If all objects of a specific kind do not form a set, we shall occasionally say that they for a "totality." Thus we may speak about the totality of sets, totality of all objects, totality of all n-tuples, etc treating the term "totality" as an informal term. There are axiomatic systems of set theory in which the concept of totality is opposed to that of set. In such systems totalities are usually referred to as "classes" rather than "totalities" (using one term or another is a matter of linguistic preferences).

1.7. Russell's Paradox

The question of which conditions of the form $\Phi(x)$ can be used to define a set was one of the most difficult questions examined by set theorists. It was undertaken and widely discussed at the beginning of twenty century in response to paradoxes invented by Bertrand Russell. One of them is the following. Define $\mathbf{N} = \{x|\ \mathbf{set}(x)$ *and* $x \notin x\}$. Of course, there are sets which are not their own elements; one may call them ***normal***. If there are normal sets, so there must be the set **N** of all normal sets, mustn't it? No. No set meets the condition imposed on elements of **N**. Indeed, if **N** were a set then either $\mathbf{N} \in \mathbf{N}$ (ignore Non-Reflexivity Principle for the time being) or $\mathbf{N} \notin \mathbf{N}$. But $\mathbf{N} \in \mathbf{N}$ implies that $\mathbf{N} \notin \mathbf{N}$ and $\mathbf{N} \notin \mathbf{N}$ implies that $\mathbf{N} \in \mathbf{N}$. So whichever assumption one selects, one arrives at contradiction.

1.8. Is there a set that has no elements?

A set that has no elements is called ***empty***. Can we safely (without running into inconsistencies) assume that there is an empty set? Quite a different question is of whether the idea of the empty set is consistent with the common-sense idea of a set. The present status of mathematical knowledge do not gives any reason for questioning the following

> **1.7.1. THE EMTY SET PRINCIPLE**: *There is a set such that no object is its element.*

(Use $\exists$ as the symbolic notation for *there is* and $\wedge$ as that for *and*. Then the symbolic variant of 1.7.1 is $\exists x(\mathtt{set}(x) \wedge \forall y\, y \notin x)$.

The positive answer to the question of whether there is an empty set has not been derived from an analysis of the common-sense idea of the concept. Rather it was motivated by practical considerations similar to those that resulted in introducing the concept of zero. Without the concept of zero the language of arithmetic is to poor to handle some questions. For instance, it does not allow one to answer to the question "how many people are in the room?" if nobody occupies that room. As we shall see this in the next Chapter, much the same can be said about the language of set theory without the concept of the empty set. This one we are going to close by introducing the symbol of the empty set.

Before we do this let us observe the following. As long as we do not question 1.7.1, we can bed sure that there is an empty set. Can we be sure that there is only one? Yes, for if both A and B is empty, then the objects being elements of either set are exactly the same: none.

Remove one by one all elements from $\{a_1,a_2,...,a_n\}$. The outcome will be the empty set $\{\}$. This symbol is applied by computer scientists. Mathematicians prefer to denote the empty set by $\varnothing$. We shall use the latter.

2. A Formal Analysis of a Discourse

2.1. Three Components of a Discourse

Our interest in logic concerns the role it plays in any rational (at least in the sense that its participants do not change the rules of the discourse at will) ***discourse*** whatsoever, meant to be an exchange of ideas on a fairly well defined subject matter. Propositions that are produced as outcomes of the discourse might form various ***belief systems*** representing alternative solutions to the discussed issues. Thus e.g. set theory is a discourse whose subject matter is the meaning of the term set. Various axiomatic systems of set theory are belief systems representing alternative standpoints taken by participants of the discourse. Of course, various informal discourse in which we engage on various occasion are not as well defined (and thus not as rigorous) as mathematical theories.

Every discourse at every its stage is determined by: (1) the objects that are examined, (2) the ***subject matter*** of the discourse meant to be the set of questions that are to be answered; (3) The ***assumptions*** of the discourse meant to be the set of propositions that the participants of the discourse treat as unquestionable. Objects dealt with within a discourse might be of any kind whatsoever, notably physical objects, mental objects (emotions, feelings, beliefs) and intencional objects (values, norms, abstract concepts). The objects examined in set theory are various set concepts rather than just sets. Actually they often divide into different "sorts" or "universes" of which usually one is "main" while the other "supplementary" (cf. XXX).

As a discourse develops every of three parameters that determine the discourse may change. However, as a rule, a ***formal analysis*** of a discourse, meant to be an analysis that can be carried out with the help of concepts and techniques offered both by logic and mathematics (notably set theory) concerns the discourse at its specific stage. Logical investigations into dynamics of belief systems is a relatively area of inquiry, and covers only some aspects of discourse dynamics.

2.1. The Universal Set

One of the key concepts we are going to use in the discussion that follows is that of a "subset" of a set.

> **2.2.1. DEFINITION OF A SUBSET**: *A set A is a* ***subset of*** *a set B (is* ***included in*** *B), in symbols* $A \subseteq B$, *iff every element of A is an element of B.*

One may formalize this definition as follows: $A \subseteq B \equiv_{df} \forall x\ (x \in A \rightarrow x \in B)$. The subscript *df* added to the equivalence connective $\equiv$ indicates that the equivalence holds true "by definition" (by definition the left part of the equivalence is another way of saying that what the right part says, see also XXX).

If $A \subseteq B$ and $A \neq B$ we write $A \subset B$ and say that A is a ***proper subset of*** (is ***properly included in***) B.

Suppose **U** is the set of all examined objects within a discourse. Such a set (we shall always take for granted that it is not empty) is called the ***universe*** of the discourse (a non-logical term for such a set is "population"). Very often the substantial part of analyses carried out within a discourse concerns various relations that connect various subsets of the universe. Needless to say that of special interest is those subsets that can be defined as sets of all objects in **U** that meet a condition $\Phi(x)$ stated in the language of the discourse. The notation for such a set is $\{x \in \mathbf{U} \mid \Phi(x)\}$). Some of such subsets might happen to be empty and thus, if the concept of an empty set were not included into the language of set theory, prior to using the notation we have introduced it would be necessary to examine whether there is a set to which the notation supposedly applies. This apparently inessential restriction would both complicate and impoverish the language of set theory considerably.

2.2. Algebras of Sets

Though the concepts we are introducing in this section are rather "technical." On the other hand the range of their applicability, in particular in formal analyses of discourses, is rather large.

2.5.1. SELECTED EXISTENTIAL PRINCIPLES: *For every sets A and B there are the following sets:*

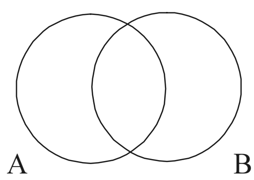
- *The **relevant complement** of B with respect to A, in symbols $A - B$, i.e. the set of those elements in A which are not in B.*
- *The **intersection of** A **and** B, in symbols $A \cap B$, i.e. the set of objects which are both in A and B. (If $A \cap B = \varnothing$, the sets A, B are said to be **disjoint**.)*
- *The **union of** A **and** B, in symbols $A \cup B$, i.e. the set of objects which are in either A or in B (at least in one of these sets).*
- *The **power set** of A, in symbols 2^A, i.e. the set of all subsets of A.*

We trust the reader to state the clauses of 2.5.1 in the formalized manner.

If A is a subset of a universe **U**, then instead **U** – A one writes $-A$. The set $-A$ is called ***the complement of*** A (***with respect to*** **U**). Note that given any power set $2^{\mathbf{U}}$ the outcome of applying operations $\cap$, $\cup$, – to elements of $2^{\mathbf{U}}$ is again an element of $2^{\mathbf{U}}$. The system ($2^{\mathbf{U}}$, $\cap$, $\cup$, –) which the power set $2^{\mathbf{U}}$ forms along with the operations $\cap$, $\cup$, – is an example of "algebra of sets" (a subset of a power set defines an ***algebra of sets*** if the outcomes of operations $\cap$, $\cup$, – applied to elements of that subset are its elements again). Note that for every universe **U**, the system ({U, $\varnothing$}, $\cap$, $\cup$, –) is an algebra of sets, known as ***two-element Boolean algebra***.

2.3. Venn diagrams

The interiors of the circles A, B below may serve as a graphic representation of sets A, B.



Then: (1) The left "half-moon" is $A - B$; (2) the middle segment (the lens) is the intersection $A \cap B$; (3) The right "half-moon" is $B - A$; (4) All the three segments jointly represent the union $A \cup B$.

In order to indicate that a segment is empty, we shall shade it. In order to indicate that it is not we shall mark it with +. Thus, if a segment is neither shaded nor contains +, the question of whether it is empty or not is left open.

2.4. Informal vs. Formalized Discourses

We shall distinguish between formalized and informal discourse (thus formalized and informal belief systems, in particular). To ***formalize*** a discourse is to formalize both its grammar (the rules that govern the way in which the sentences relevant for the discourse are formed) and its semantics (the rules that the participants of the discourse observe when they interpret the sentences of which the discourse is composed).

To formalize the grammar (syntax) of a discourse is to set rules that define rigorously how the sentences that might appear as elements of the discourse should be formed. For instance, among others the following sentences: (1) *x is an element of A*, (2) *x is in A*, (3) *x is one of the objects of which A is composed* might serve as informal statements to the effect that x is an element of A. Typically enough (though not necessarily) formalized languages take symbolic form and thus, in particular, $x \in A$ is customarily applied as the formalized counterpart of the above sentences.

To formalize the semantics of a discourse is to set the rules that prevent one from assigning a wrong interpretation (meaning) to expressions applied in the discourse. An interpretation which the participants of the discourse do not consider wrong will be called ***acceptable***. The notion of an acceptable interpretation is going to be discussed on several occasions (see especially XXX). Its present explication is tentative.

2.5. Primitive vs. Auxiliary Terms

The terms whose meanings are supposed to be determined, though not necessarily in a complete way, by postulates and definitions of a belief system are referred to as ***primitive*** terms of that system. All the remaining terms that appear in the language of such a system will be called ***auxiliary***.

Under this stipulation, the expressions such as e.g. *not*, *or*, *if...then*, *for all*, thus expressions from the list of logical terms (cf. 1.4.1), are typical auxiliary terms. They need not be the only terms of this kind. Thus e.g., various mathematical theories are based on other mathematical theories in the sense that the meaning of their primitive terms is explicated with the help of terms known from mathematical theories established earlier. In other words, the primitive terms of the latter serve as auxiliary terms of the former. For instance the primitive terms of the theory of real numbers serve as auxiliary terms of probability theory, on the other hand primitive terms of probability theory serve as auxiliary terms of decision theory. Of course, there are theories (set theory is one of them) whose only auxiliary terms (except those that can be introduced by definitions, cf. the next Section) are logical constants.

2.6. Nominal definitions, postulates and meaning stipulations

A special category of auxiliary terms are defined terms. A properly defined term should be eliminable in the sense that every proposition in which it appears can be proven to be equivalent to a proposition in which it does not. A definition that introduces a new term as a replacement of either an expression or a grammatical construction that the language of the discourse already contains (and thus meets the eliminability condition in a direct way) is called ***nominal***.

It is customary to announce that a statement is a nominal definition by furnishing either identity symbol = or equivalence symbol ≡ (depending on which of them was applied as the key symbol of the definition) with subscript *df* (e.g. $\varnothing =_{df}$ *the set that has no elements* or $x \notin y \equiv_{df} \neg(x \in y)$.

In some contexts, "to define" means to produce a statement that determines the meaning of the defined word in the unique way, provided that the meaning of all the remaining terms appearing in the definition are fixed. A definition that is meant to be a unique characteristic of the meaning of a word is called ***real***. In fact, every nominal also is also real; it defines the meaning of the new term as the same as that of the expression the new term replaces. The converse may happen not to be true. In the original language there might not be any expression that has the same meaning as that assigned to the term introduced by a real definition. For the time being, we may restrict our comments on the concept of a definition to those given above (for more on that concept see XXX).

3. Logic of categorical sentences

3.1. Categorical Sentences

Let a universe **U** be given. Then, the elements of **U** are called ***individual objects*** (***individual***, for short) and names for them are called ***individual names***. Every individual name is either a ***description*** (e.g. *the capital of the US*, *the smallest natural number*) or a ***proper name*** (e.g. *Bill Clinton, 0)*. The former "describe" the object to which they refer, the latter only "name" it. The above distinction between descriptions and proper names is informal and applies to informal languages. In a formalized language it should be defined in terms of semantic categories (see XXX).

A ***general name*** (e.g. *town*, *president of the US*, *odd number*) is a name intended for singular objects of a specific kind (or "category"), so nothing in the "content" of the name indicates to how many objects it refers (the ***content*** of a name is the set of criteria one uses in order to judge whether the name applies to a given object or not). It may happen thus that a general name, e.g. the city in Poland with more than millions inhabitants, refers to exactly one object. This does not make it an individual name, however.

A ***categorical sentence*** is a sentence that can be paraphrased to take one of the following forms (**S** and **P** stand for general names): (1) *Every* **S** *is* **P***,* (2) *Some* **S** *are* **P***,* (3) *No* **S** *is* **P** or (4) *Some* **S** *are not* **P**; in symbols: **S** *a* **P**, **S** *i* **P**, **S** *e* **P** and **S** *o* **P**, resp. The symbols *a, e, i,* and *o* will be called ***categorical quantifiers***.

Examples: *All cats are smart (Every cat is smart), Some dogs hate cats (Some dogs are cat haters).* The logic of categorical sentences was studied by Aristotle.

3.2. Formalized Categorical Language

By a formalized categorical language we shall mean a language that meets the following three requirements. (i) It does not contains other categorical sentences other than that of the form **S** *a* **P**, **S** *i* **P**, **S** *e* **P** or **S** *o* **P** and their negations. The latter are formed with the help of the symbol ¬; (ii) All general names that appear in the vocabulary of the language refer to elements of the same fixed universe; under the convention we are adopting they will be written in bold courier and hyphenated, if they include more than one word, e.g. **`black-cat`**, **`president-of-the-US`**.

(iii) Besides the logical constants and general names mentioned above, no other terms (individual names including) are in the vocabulary of the language.

Example. The sentence **`cat`** *a* **`smart`** is formalized (one can easily define a formalized language in which it can be formed). The sentences *every cat is smart*, *all cat are smart* and many others are its informal counterparts.

3.3. Deduction Statements

Given any set of sentences X and any sentence α write $X \models \alpha$ in order to state that α ***is a logical consequence*** of (sentences in) X. Alternatively $X \models \alpha$ reads "α follows logically from X." One also may say that "X logically entails α".

Whenever for a specific X and a specific α one states that $X \models \alpha$, the sentences in X will be called the ***premises*** of the expressed proposition and α its ***conclusion***.

Some notation. Rather than writing $\{\alpha_1,\alpha_2,\ldots, \alpha_n\}\models \alpha$, we shall prefer to write $\alpha_1,\alpha_2,\ldots,\alpha_n \models \alpha$. Moreover the following two conventions will be adopted: (1) $\models \alpha$ abbreviates $\varnothing \models \alpha$ and (2) $X, \alpha_1,\alpha_2,\ldots,\alpha_n \models \alpha$ abbreviates $X \cup \{\alpha_1,\alpha_2,\ldots, \alpha_n\} \models \alpha$.

Example.

[D] **`cat`** *a* **`smart`**, **`cat`** *i* **`black-cat`** $\models$ **`smart`** *i* **`black-cat`**

Does deduction statement [D] holds true? In other words: does the sentence **`smart`** *i* **`black-cat`** follows from the sentences **`cat`** *a* **`smart`** and **`cat`** *i* **`black-cat`**.

In order to handle questions of this kind the notion of logical consequence has to be defined. The following approach to this issue is based on ideas s et by Aristotle.

3.4. Logical Entailment

Let $\alpha_1, \alpha_2,\ldots, \alpha_n \models \alpha$ be a deduction statement. Replace general names that appear in $\alpha_1, \alpha_2,\ldots, \alpha_n$. α by variables **S**, **P**, **M**, … representing arbitrary general names (such variables are often called ***meta-variables***, cf XXX). Take care of replacing the same names by the same variables and different names by different variables. Then the outcome of the replacement is a ***deduction schema***. The deduction statement $\alpha_1, \alpha_2,\ldots, \alpha_n \models \alpha$ is said to be its ***instance*** (or to "fall under" it) Given a sentence β, the outcome of such replacement is called a logical schema of β.

Example. If in [D] **`cat`** is replaced by **M**, **`smart`** by **S**, and **`black-cat`** by **P** the resulting deduction schema of [D] will be

[Darii] $\mathbf{M}\ a\ \mathbf{S}, \mathbf{M}\ i\ \mathbf{P} \models \mathbf{S}\ i\ \mathbf{P}$

3.4.1. LOGICAL VALIDITY OF A DEDUCTION SCHEMA (A TENTATIVE DEFINITION): A deduction schema is ***logically valid*** iff for no its instance the premises are true and the conclusion is false.

The basic idea of logical analyses carried out by Aristotle was that $\alpha_1, \alpha_2, \ldots, \alpha_n \models \alpha$ holds true iff the logical terms that appear in sentences $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ rule out that $\alpha_1, \alpha_2, \ldots, \alpha_n$ might be true and yet α false. One way to make this definition precise is the following:

3.4.2. LOGICAL CONSEQUENCE: Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ be formalized categorical sentences. Then $\alpha_1, \alpha_2, \ldots, \alpha_n \models \alpha$, iff $\alpha_1, \alpha_2, \ldots, \alpha_n \models \alpha$ is an instance of a logically valid deduction schema.

In virtue of 3.4.2, in order to show that α is not a logical consequence of $\alpha_1, \alpha_2, \ldots, \alpha_n$, (in symbols $\alpha_1, \alpha_2, \ldots, \alpha_n \not\models \alpha$) it suffice to ***invalidate*** the relevant deduction schema (i.e., to find out an instance of the schema whose premises are true and the conclusion is false). Thus e.g. (as has already been argued by Aristotle) the schema

$$\mathbf{M}\ a\ \mathbf{S}, \mathbf{P}\ e\ \mathbf{M} \models \mathbf{S}\ a\ \mathbf{P}$$

is invalidated by: **`creature`** (**S**), **`man`** (**M**), **`horse`** (**P**). On the other hand, no substitution invalidates [Darii] and hence the sentence **`smart`** *i* **`black-cat`** follows from the sentences **`cat`** *a* **`smart`** and **`cat`** *i* **`black-cat`**. The question is how one can know that no substitution invalidates [Darii].

3.5. The Laws of Logic

The chief task of logical investigations is to find out which sentences follow from which. Thus, in the basic sense of the word, a ***law of logic*** is a statement to the effect that given a set of sentences along with a sentence, both being of a specific kind (e.g. being instances of specific schemata), the latter is a logical consequence of the former. In other word a law of logic is a statement to the effect that for some specific X and some specific α, $X \models \alpha$ obtains. For instance, as one easily agrees, from any premises of the form **S***a***M**, **M***a***P** it follows conclusion of the form **S***a***P**, thus the statement we have just made is a law of logic. Note that to state that thus described relation between premises and conclusion holds true is to state that the deduction schema

[Barbara] **S***a***M**, **M***a***P** |= **S***a***P**

is logically valid.

3.6. Abstract Categorical Languages

The Aristotelian logic is hand in glove with the following world view, one might call it ***Aristotelian***. The world is a universe of individual objects that in some natural way divide into different "categories". Thus to know the structure of the world one must know how different categories of individual objects are related to one another. The categorical quantifiers provide the basic tools for such analyses.

The Aristotelian world-view does not provide a good framework for forming rich and subtle enough knowledge about the world. One of the reasons is that the world such as we able to see it divides into various overlapping universes rather than a single one. This explains why some grammatically correct categorical sentences (e.g. *Every stone is hungry* or *Some hungry objects are circles*) are semantically meaningless. Another is that the Aristotelian idea of a category is rather enigmatic. Is category the same as a set of individual objects? And if it is not, which sets of individual objects form a category and which do not? These two points seemingly of minor significance are both relevant to our present considerations.

The Aristotelian idea of validity of a deduction schema is that a deductive schema is valid if and only if it cannot be invalidated by substitutions one might form using general names available in the commonly used language. The following alternative approach is suggested by Tarski's analysis of consequence operation (XXX).

Select an universe **U**. Call a categorical language an **U**-***language*** if all its general names are names of objects in **U**. We are now in a position to improve definition 3.4.1.

> **3.6.1. LOGICAL VALIDITY OF A DEDUCTION SCHEMA (AN IMPROVED DEFINITION)**: A deduction schema is ***logically valid*** iff for no universe **U** and no **U**-language, there is an instance of that schema such that its premises are true and its conclusion is false.

The basic advantage of definition 3.5.1 is that instead of examining validity of deductive schema by appealing to our intuitive knowledge encoded in everyday language we might now use the concepts and techniques of set theory.

4. Set-theoretic Analysis of Aristotelian Logic

4.1. Meta-language

Let us begin with some rather obvious explanations. In both this chapter and the previous one we have been discussing various formalized categorical languages. The language in which such a discussion is carried out is called ***meta-language***, and the language discussed is called ***object language***. Expressions that appear in meta-language are often called meta-expressions (e.g. meta-variables or meta-sentences). Those which belong to object language might be called object-expressions.

The basic criterion for distinguishing expressions of the object language from those of meta-language is that we ***use*** the former to say something about some "non linguistic" states of affairs, while the latter are applied to ***refer*** to (to state something about) the former. Thus e.g. we refer to the sentence "**`cat`** *a* **`smart`**" and indirectly we refer to its informal counterpart "*Every cats are smart*" whenever we state anything about that sentence, e.g. that it is true. On the other hand we may either in object language or in meta-language use any of these two sentence say "*Every cats are smart*" to state that cats are smart. Since both referring to a sentence and using it might be confusing, we will not use formalized sentences in meta-language but only refer to them. This cannot be the case of informal statements however. In meta-linguistic considerations they will be both referred and used.

Another source of confusion might be the practice of using the same symbols as object variables and meta-variables. We shall turn back to this point at the right time.

4.2. Set-theoretic Analysis of Categorical Statements

Given a general name **S**, denote by [**S**] the set of all objects to which the name refers. It will be referred to as the ***extension*** of **S**. Thus in particular if $\mathbf{S}_X$ is a general name of a **U**-language then $[\mathbf{S}_X] = X$.

One way to understand the categorical sentences is the following:

$[a_\in]$ $\qquad \mathbf{S}\ a_\in \mathbf{P} \equiv_{df} [\mathbf{S}] \subseteq [\mathbf{P}]\ (\equiv_{df} \forall x(\mathbf{S}(x) \rightarrow \mathbf{P}(x)))$

$[i_\in]$ $\mathbf{S}\ i_\in \mathbf{P} \equiv_{df} [\mathbf{S}] \cap [\mathbf{P}] \neq \varnothing\ (\equiv_{df} \exists x(\mathbf{S}(x) \wedge \mathbf{P}(x)))$

$[e_\in]$ $\mathbf{S}\ e_\in \mathbf{P} \equiv_{df} [\mathbf{S}] \cap [\mathbf{P}] = \varnothing\ (\equiv_{df} \neg\exists x(\mathbf{S}(x) \wedge \mathbf{P}(x)))$

$[o_\in]$ $\mathbf{S}\ o_\in \mathbf{P} \equiv_{df} [\mathbf{S}] - [\mathbf{P}] \neq \varnothing\ (\equiv_{df} \neg\forall x(\mathbf{S}(x) \rightarrow \mathbf{P}(x)))$

With these definitions being accepted, the question of validity of deduction schema becomes a question of validity of a statement that concerns relations between sets. Indeed, consider for instance the schema [Darii]. In order to invalidate [Darii] one must find a universe **U** and three its subsets X, Y and Z such that $[\mathbf{M}] = X$, $[\mathbf{S}] = Y$ and $[\mathbf{P}] = Z$ and moreover $X \subseteq Y$, $X \cap Z \neq \varnothing$ and nevertheless $Y \cap Z = \varnothing$. Yet, as one might fairly easily prove, the following is a theorem of set theory: *If $X \subseteq Y$ and $X \cap Z \neq \varnothing$ than $Y \cap Z \neq \varnothing$*. If so, [Darii] cannot be invalidated by any substitution in any **U**-language whatsoever.

4.3. Logical Foundations of Logic

The above presented proof of validity of [Darii] provokes the following question. The proof that involves set-theoretic considerations involves also logic underlying set theory. Does such an argument prove anything or rather is a typical example of vicious circle?

The question deserves our attention. We surely cannot prove logical validity of a deduction schema without using any "logic". The right policy is the following. Call a mode of reasoning that never results in arriving to false conclusion from true premises ***reliable***.

At each stage of development of our knowledge we have some ideas of reliable modes of reasoning. A natural laboratory for discovering them is mathematics. Although the Aristotelian logic was in no direct way motivated by mathematical analysis, Greek philosophers fully appreciated the role of mathematics in setting standards of correct and precise reasoning. On the other hand, Greeks were able to identify only some of modes of reasoning applied in mathematics and state them only in very general terms. Discovering how the mathematicians reason took very long time; it was completed only at the beginning of twenties century.

Once we establish reliability of a mode of reasoning we are entitled to use them in all arguments, the proofs of logical validity of deduction schemata included. Thus "theoretical" logic (logic meant to be a system of logically valid deduction schemata) is based on "practical" logic – the logic of reliable modes of reasoning applied by competent mathematicians, also those who never learned "theoretical" logic. If we look at the thing in this way, there is nothing inappropriate in using set-theoretic arguments for proving logical of validity of logical schemata.

It might happen (the Russell paradox is an example) that apparently reliable mode of reasoning results yields contradiction. In each such case the logical foundations of mathematics (and consequently all sciences based on it) should be reexamined. In fact, no matter how reliable seems to be a way of reasoning we never can be sure it cannot be a source of antinomies. No skeptical conclusion (any thesis of impossibility of arriving at reliable knowledge) can be derived from this observation. The open question is, however, of whether the continuous process of improving the standards of reliability might eventually be halted by conceptual problem we are not able to overcome. If this happens (we cannot know if it does), the process of knowledge formation will stop and our past achievements will be put in question.

4.4. Are Laws of Aristotelian Logic Valid?

By a ***(deduction) law of logic*** we shall mean a true statement to the effect that a deduction schema is valid. Curiously enough, under the set-theoretic approach presented in 3.6 some of the deduction schemata identified by Aristotle as logically valid can be invalidated. For instance the following two (known as ***subordination laws***):

(A1) $\mathbf{S}\ a\ \mathbf{P} \models \mathbf{S}\ i\ \mathbf{P}$,
(A2) $\mathbf{S}\ e\ \mathbf{P} \models \mathbf{S}\ o\ \mathbf{P}$

Consider (A1). If **S** is an ***empty name*** (i.e. its extension is empty) then $[\mathbf{S}] \subseteq [\mathbf{P}]$ (to say that if $x \in [\mathbf{S}]$ then $x \in [\mathbf{P}]$ is tantamount to saying that there is no x such that $x \in [\mathbf{S}]$ and $x \notin [\mathbf{P}]$). Thus $\mathbf{S}\ a_{\in}\ \mathbf{P}$. And yet $[\mathbf{S}] \cap [\mathbf{P}] = \varnothing$, which means that $\mathbf{S}\ i_{\in}\ \mathbf{P}$ is false. Consequently, the deduction schema (A1) applied to a true sentence (e.g. **`mermaid`** *a* **`blond`**) may yield a false one. E.g. in virtue of that schema from true sentence **`mermaid`** *a* **`blond`** one might derive as its logical consequence **`mermaid`** *a* **`blond`**, which is false. We trust the reader to check that the case of (A2) is fully the same. Every empty name **S** invalidates the schema.

4.5. Paradoxes of empty names

To begin with note that in order for a categorical language to enable one to invalidate (A1) and (A2) the language must contain at least one empty name. Consequently, the Aristotelian logic remains sound if the scope of its applicability is restricted to languages with no empty names. The question is: are there any good reasons to postulate such a restriction?

One of such reasons is that from the intuitive standpoint both the sentence **`mermaid`** *a* **`blond`** and the sentence **`mermaid`** *e* **`blond`** (in fact any two sentences instantiating schemata **S** *a* **P**, **S** *e* **P**) cannot be simultaneously true; it does not make sense to maintain at the same time that all mermaids are blond and no mermaid is blond. As a matter of fact, the statement to the effect that these two sentences cannot be simultaneously true is one of the laws of the Aristotelian logic. It is not clear whether Aristotle was aware of problems that would arise if his logic were applied to empty names.

4.6. Was Aristotle right?

Some animals are predators and some are not. But of these two facts the latter is not a logical consequence of the former. One might imagine a world in which all animals are predators. If some animals are terrestrial and some are reptiles, then one might wonder if there are both terrestrial reptiles and non-terrestrial reptails. Consider a discourse for which these questions are relevant. Is their logical analysis possible with the help of Aristotelian logic? Not quite so. The idea of non-predator involves complement operator *non* and the idea of terrestrial reptile involves (though in the hidden form) the intersection operator *and*. The idea of non-terrestrial reptiles involves both. Recall that these two operators are known from the set-theoretic analyses in which they are respectively denoted by $-$ and $\cap$.

Though (as a matter of fact for no substantial reasons) set theory is not treated as a part of logic, the role which set concepts play in a discourse is much the same as that of logical concepts; they serve as basic tools for analysis of how the extensions of the primitive terms of the discourse are related to one another. In order for such analysis to be possible the discourse should be formalized in a language which contains the two constants. Of course, that can be easily done and of course the resulting language will be essentially richer that the categorical language in its original form. Since the constants $-$ and $\cap$ apply to names, the Aristotelian logic expanded by them is occasionally called logic of names.

Let us now turn back to the question that ends the previous chapter. Should the empty names be ruled out from formalized languages? The case when one claims both that all mermaids are not blond and none is blond, might be ignored as one that concerns unreal world. But what about someone claim that all policemen that enter the room were drunk in the situation when no policemen enter the room. Do we really might agree to treat the sentence *All policemen that enter the room were drunk* as true just because the sentence is about non-existing policemen?

Yes, the semantics of statements that involve empty names allows one to produce true and counterintuitive statements. So perhaps one should ban using empty names in a rigorous discourse. On the other hand there is something annoying in the fact that given names S and P, one neither can form the name $-S$ nor $S \cap P$, unless one verifies in advance that the name in question is not empty. So the choice is between counterintuitive semantics and unduly restrictive grammar. This problem is much the same as that concerning the empty set (cf. XXX). If the subject of logical analysis is an everyday discourse we might prefer to ban using empty names. If the discourse concerns some sophisticated formal issues, we may prefer avoid unduly restrictive grammatical restrains. Thus, e.g., the formalized languages of mathematical considerations allow one for producing statements that both apparently deny one another (e.g. *All prime number different from 7 divide by 7* and *None prime number different from 7 divide by 7*) and are simultaneously true for otherwise it would be necessary to ban using empty names (e.g. *a prime number different from 7 that divides by 7*).

4.7. Some logical relations definable in terms of entailment

Two sentences α and β are said to:

- ***Exclude one the other*** iff $\alpha \models \neg\beta$
- ***Complement one the other*** iff $\neg\alpha \models \beta$
- ***Contradict one the other*** iff they are both exclusive and complementary.

(Note that: (1) $\alpha \models \neg\beta$ iff $\beta \models \neg\alpha$, and (2) $\neg\alpha \models \beta$ iff $\neg\beta \models \alpha$)

5. Predicates

5.1. Some Examples

The following is an example of a proposition that cannot be expressed in the form of a categorical sentence: *For every prime number x there is a prime number y greater that x.* Actually, the proposition can be expressed in categorical language, but to do this one has use infinitely many categorical sentences of the form **`prime-number`** *o* **`prime-number-that-is-not-greater-than-x`**, where **`x`** should be replaced by the individual name of a prime number.

A rather informal statement *Regardless of how rich is someone, sooner or later somebody will be richer than her/him* has a similar formal (though not exactly the same) formal structure than the previous one. It cannot be formalized with the help of a finite number of categorical sentences either.

In spite of what was believed by many eminent philosophers still at the beginning of twenties century, hardly any discourse can be adequately presented as a finite sequence of categorical statements. As a rule, logical analysis of a discourse requires both richer descriptive means and stronger logical tools than those provided by the idea of categorical language combined with that of Aristotelian logic.

The languages we are going to examine beginning from this Chapter are known as "first order predicate languages with identity." Their descriptive means do not reduce to general names and their logical means are the same as those applied in mathematical reasoning. Even though one should not expect that every discourse can be formalized within first order predicate language (cf. XXX) both theoretical and practical significance of these languages is enormous.

5.2. One-Sorted Relations vs. One-Sorted Predicates

Rather than carrying our discussion in a fully general way, let us assume that we want to analyze the formal structure of a discourse whose subject matter concerns a population (universe) **H** of people. The examined issues determine the language of the discourse – both its ***descriptive*** (i.e. directly related to the subject matter of the discourse) and its auxiliary part. Besides general names, the descriptive vocabulary of the discourse might include some ***predicates*** i.e. terms that denote relations. The predicate **`richer-than`** is an example of a ***binary predicate***, i.e. a one that stands for a binary (holding between a couple of individuals) relation. The predicate **`son-of-and`** is an example of a ternary predicate – a predicate that stands for a relation that is defined for triples of objects, etc. It will be convenient to treat general names as ***unary predicates*** and to treat the sets of individuals they denote (their extensions) as ***unary relations***.

Switching from ordinary notation to formalized one (writing **`richer-than`** instead of *is richer than*, **`son-of-and`** instead of *is a son of ... and...*) should not reduce to typography. It should be accompanied by semantic analyses that explicate the meaning in which the formalized expressions are applied in the discourse. Thus e.g. of two eco-systems one might be richer than the other, but of course, the term *richer*, in a discourse that concerns people (more specifically those who form the population **H)** does not mean the same as it means in a discourse that concerns eco-systems (histories, to exhibitions, etc.). Actually it may change from one population to another.

The predicates and relations we have discussed above are "one-sorted" in the sense we are going to explain in the next Section.

5.3. Many-Sorted Relations and Many-Sorted Predicates

A discourse might require examining relations that hold between objects of two different "sorts" (***two-sorted relations***), say between the examined objects that form the familiar universe **H** and objects that form a "supplementary" universe **T** of, say, time instances. Also, it might require examining relations that connect objects belonging to three different "sorts" (***three-sorted relations***), say **H**, **T**, and a "supplementary" universe **L** of selected "locations," etc.

An example of a binary two-sorted predicate (thus a predicate that denotes a binary two-sorted relation between persons and time instances) is **`born-on`**. The predicate ^{1}**`met`**^{2}**`-at`** (the superscripts 1 and 2 indicate that the predicate applies to two persons) is an example of two-sorted ternary predicate. An example of a three sorted "4-ary" predicate is **`moved-from-to-at`**.

5.4. Atomic Wffs

In order to form a sentence with the help of a n-*ary* predicate, one should supplement the predicate with *n*-tuple (*n* element sequence) of individual names of the right kind. Thus a sentence formed by *n*-ary predicate **`P`** is of the form **`P`**$(x_1, x_2, \ldots, x_n)$. Note that we describe here the sentences formed by the predicate, thus the variables x_1, x_2,…, x_n are not object variables (cf. XXX) but meta-variables representing individual names.

Example. The first term that follows after the predicate **`moved-from-to-at`** should be an individual name of a person (e.g. `Thomas-Mann`), the second and the third should be both individual names of some "locations" (e.g. `Germany` and `USA`), the last one should be the date (e.g. `02.09.1939`). The sentence formed in the way indicated by the examples and presented in an abbreviated version is

`moved-`…(`Mann`, `Germany`, `USA`, `02.09.1939).`

A term that complements an expression is called its ***argument***. A sentence formed by predicate completed by arguments each being an appropriately selected individual name is called ***atomic***. If any individual names are replaced by variables, the formula resulting from an atomic sentence is an ***atomic sentential function***. Atomic sentences along with atomic sentential formulas form the set of all ***atomic well-formed-formulas***, ***wff's*** for short.

5.5. Extensions of Predicates

Recall that if **`P`** is a general name then [**`P`**] stands for its extension, i.e. the set of individual objects (elements of the defined in advance universe **U**) to which **`P`** refers. The above will be generalized as follows. Given a n-ary predicate **`P`** we shall by the ***extension*** [**`P`**] of **`P`** we shall meant the set of all n-tuples of individual objects of which the predicate is true (one may truly assert the predicate of each of them). Assume that the universes of the analyzed discourse are the familiar **H**, **E** and **L**.

Examples: (1) [**`younger-than`**] is the set of all pairs (x, y) of persons in **H** such that x is younger than y; (2) [**`born-on`**] is the set of all couples (x, y) such that x is in **H**, y is in **E** and x was born at the time of y; (3) [^{1}**`moved-from`**^{2}**`-to`**^{3}**`-before`**^{4}**`-moved-from`**^{5}**`-to`**^{6}] (the superscripts indicate the places where the arguments of the predicate should be placed) is the set of 6-tuples (x_1, x_2, …, x_6) such that x_1 and x_4 are in **H** while x_2, x_3, x_5, and x_6 are in **L** and x_1 moved from x_2 to x_3 before x_4 moved from x_5 to x_6.

Is the extension [**`P`**] of **`P`** the same as the relation the predicate denotes? The common-sense of the term "relation" does not allow one to answer to this question conclusively, so in order to answer to it one has to explicate the meaning of that term one way or another. From now on by a n-ary relation we shall mean a set of n-tuples of the defined in advance kind (see the nest Section) in particular n-tuples that form the extension of a predicate. Thus we have settled the question that opens this paragraph in positive.

5.6. Extensions of Predicates

Can the term extension and the corresponding notation be applied to individual names? This is a verbal question, i.e. one that one might decide as one wish; there no clear criteria of good solution to it. We decide it in positive. Thus given a name `p` we shall denote by [`p`] the unit set whose only element is the referent of the name.

Incidentally, if a name is considered to be a term of a formalized language we shall indicate this fact by using courier non-bold characters.

5.7. Set-Theoretic Concept of a Relation

Let ($\mathbf{U}_1$, $\mathbf{U}_2$, … ,$\mathbf{U}_n$) be a sequence of non-empty sets (recall that, the elements of this sequence need not to differ one from another). In set theory the symbol $\mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n$ stands for the set of all n-element sequences, (x_1, x_2, …, x_n) such that $x_1 \in \mathbf{U}_1$, $x_2 \in \mathbf{U}_2$, …, $x_n \in \mathbf{U}_2$. Te set thus defined is referred to as the ***Cartesian product*** of $\mathbf{U}_1$, $\mathbf{U}_2$, … ,$\mathbf{U}_n$.

A *n-**ary relation*** in an abstract set-theoretic sense of the word is a set **R** such that $\mathbf{R} \subseteq \mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n$, for some non-empty sets $\mathbf{U}_1$, $\mathbf{U}_2$, … ,$\mathbf{U}_n$.

5.8. Wffs formed by means of predicates

Suppose [**`P`**] $\subseteq \mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n$, where $\mathbf{U}_1$, $\mathbf{U}_2$, … ,$\mathbf{U}_n$ are universes of the discourse whose lists of predicates includes **`P`**. Then

> **5.6.1a. WELL FORMED ATOMIC FORMULAS**. A formula of the form **`P`**(x_1, x_2,…, x_n) is an ***atomic wff*** (a ***well-formed atomic formula***) iff the symbols that occupy the places indicted by the meta-variables x_1, x_2, …, x_n are expressions that apply to (i.e. name or represent individual objects in the corresponding universes $\mathbf{U}_1$, $\mathbf{U}_2$, … ,$\mathbf{U}_n$.
> **b. ATOMIC SENTENCES**. A wff of the form **`P`**(x_1, x_2,…, x_n) is an ***atomic sentence*** iff no its argument is a variable.

Note the following. Before one starts forming atomic wffs, one has to decide which variables will represent which individual objects. As is customary the totality (the set if it is a set) of all objects represented by a variable will be called its ***scope***. Thus e.g. formula **`born-on`**(x, x) cannot be a well-formed formula because no person is a date on which he or she was born. The rule of using the different variables to represent objects of different kind (belonging to different universes) has been violated.

The above definitions are often stated in terms of "semantic categories" of the terms of which the formula **`P`**$(x_1, x_2,\ldots, x_n)$ is composed.

5.9. Semantic Categories of Predicates and Individual Names

The concept of a semantic category cannot be define unless the language (or a specific class languages) to which this applies is defined. Our ultimate task is to define a class of languages know as "first order predicate languages." Tentatively we might define these languages as languages whose simplest (atomic) wffs are formed by predicates in the way we have described and all the remaining wffs are formed by means of the logical constants listed in 1.4. As we are going to see, atomic wffs can also be formed by "operators" meant to be descriptive terms that denote functions (see XXX) and identity predicate.

Let us agree to use the term ***predicate language*** as a general name for all languages whose atomic sentences are formed either by predicates or by operators along with identity symbol and whose compound sentences are formed by specified in advance connectives and quantifiers. Neither connectives nor quantifiers need to be classical (cf. 1.4). Though the definition of a predicate language we have offered is rather loose, the concept will be useful.

Rather than to introduce the concept of a semantic category in an explicit manner we shall restrict ourselves to adopting the following

> **5.6.2. THE PRINCIPLE OF IDENTITY OF SEMANTIC CATEGORIES**: Two descriptive expressions ξ and ζ of a predicate language are of the same semantic category iff the result of replacing in an atomic wff one of these expressions at any of its occurrences by the other is again an atomic wff.

Note that under the above principle the following obtains: (1) two predicates **`P`** and **`Q`** are of the same semantic category iff their extensions are subsets of the same Cartesian product of universes of the discourse in which these two predicates appear, (2) two "nominal expressions" (in the discussed case, cf. also XXX, individual names or variables representing individual objects) are of the same semantic category iff the objects to which they apply (refer in the case of an individual name or represent in the case of a variable) are in the same universe.

6. Compound Well-Formed Formulas

6.1. Wffs Formed by Means of Connectives

All wffs of a predicate language divide into atomic and compound. Assume that the former have been defined. Actually the definition we have provided does not cover the case when the predicate language contains the identity symbol, cf. XXX. The way in which compound formulas are formed by means of connectives is defined as follows

> **6.1.1. COMPOUND FORMULAS FORMED BY CONNECTIVES**: Let α and β be wffs of a predicate language with classical connectives. The so are: $\neg\alpha$. $(\alpha\to\beta)$, $(\alpha\wedge\beta)$, $(\alpha\vee\beta)$ and $(\alpha\equiv\beta)$.

A more involved is the rule that governs formation of compound sentences by quantifiers.

6.2. Wffs Formed by Means of Quantifiers

To begin with, given any formula of either the form $\forall x(\Phi(x))$ or the form $\exists x$ $(\Phi(x))$, we shall say that the variable x appearing in the the formula $\Phi(x)$ is ***bound*** Note that the ***scope*** of a quantifier (i.e. the formula to which the quantifier applies) is determined by parentheses, and thus, for instance, the scope of $\exists x$ in the formula $\exists x(\Phi(x)) \vee \Psi(x)$ is $\Phi(x)$ only. Thus x is bound in $\Phi(x)$ and free in $\Psi(x)$ (unless $\Psi(x)$ contains a quantifier that bounds x in that formula also). A variable that is not bound by any quantifier is called ***free***.

> **6.2.1. COMPOUND WFFS FORMED BY QUANTIFIERS**: Let α be a wff and let x of a variable of the same discourse. Then both $\forall x(\alpha)$ and $\exists x(\alpha)$ are wffs, provided that the variable x is free in α.

The rules of using parentheses need not be exactly the same as those defined by 6.1.1 and 6.2.1. In any case, however, they should guarantee that there is only one right way of interpreting compound sentences.

6.3. Sentences vs. Propositional Functions

A wff which contains a free variable may have no definite truth value, for its truth value may change depending on which object is selected to be the one to which the variable applies. A variable that appears in an atomic wff is always free. Thus the definition 5.6.1b of an atomic sentence is consistent with the following more general:

> **6.3.1. SENTENCES**: Let α be a wff then α is a ***sentence*** iff it contains no free variable or else it is a propositional function.

6.4. Identity Predicate

Note that from the grammatical standpoint identity symbol = is a binary predicate (the fact that we write $x = x$ rather than **`is-the-same-as`**(x, x) is a matter of convention). It is not a descriptive term however; its meaning can be defined in a fully general way independent of any semantic considerations that are related to a specific discourse and its language.

For the time being we shall use identity symbol in its intuitive sense. A logical analysis of its meaning will be provided in XXX.

6.5. Functions and Operators

A natural way of using the identity symbol for forming formulas of a predicate language is the following. Suppose the relation [**`P`**] denoted by a non-unary predicate **`P`** is a function from its initial domains into the last one (the latter is often called ***counterdomain***), i.e. it satisfies the following

> **6.5.1. UNIQENESS CONDITION**: Let $[\mathbf{P}] \subseteq \mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n \times \mathbf{U}_{n+1}$. The relation [**`P`**] is said to satisfy the uniqueness condition iff for every n-tuple $(x_1, x_2,\ldots, x_n)$ in $\mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n$ **U** there is exactly one object x_{n+1} in $\mathbf{U}_{n+1}$ such that $(x_1, x_2,\ldots, x_n, x_{n+1}) \in [\mathbf{P}]$.

Examples. Let **H** denotes the population of all people. Then the predicate **`mother`** denotes a function from **H** into **H**, for everybody has exactly one mother. Suppose the set **L** of locations is defined to include all places in which somebody was born and more over it is constructed in such a way for every x in **H** there is exactly one y in **L** such that x was born in y. Then predicate **`born-in`** denotes a function from **H** to **L**.

6.6. Two Patterns of Forming Atomic Wffs

Let the extension [**P**] of predicate **P** be a function. Then instead forming the corresponding atomic formulas according to the "relational pattern" **P**(x_1, x_2,…, x_n, x_{n+1}) we may form this formula using "functional pattern" **P>**(x_1, x_2,…, x_n) = x_{n+1}, or rather **P>**(x_1, x_2,…, x_n) = x_{n+1}, where **P>** is a new term defined with the help of an old one as follows:

6.6.1. **P>**(x_1, x_2,…, x_n) = x_{n+1} iff **P**(x_1, x_2,…, x_n, x_{n+1})

Thus e.g. rather to write **born-in**(Hasek, Prague) we may write **born-in>**(Hasek) = Prague. A term that denotes a function, thus in particular every term **P>** of the kind described above will be called an ***operator***.

Needless to say that an operator may appear in a discourse as its primitive term; it need not to be introduced by a definition of the form 6.6.1. Note that in virtue of principle 5.1.2 the semantic category of an operator is always different from that of a predicate (replacing one by another in an atomic wff results in forming a formula which is "grammatically coherent" – its grammatical structure is consistent with the accepted grammatical rules). In the most general terms semantic categories of predicates symbols of relations of a specific kind, while those of operators are symbols of functions. Using symbolic notation we shall refer to operators by means of letter **F** rather than **P**.

As one may expect, given a operator **F**, the notation [**F**] will stand for the function **F** denotes. Note, however, that under the accepted definition of a function, every function is a relation. Thus in particular, for every predicate **P** that denotes a function [**P**] = [**P>**].

6.7. Atomic Formulas of First-Order Languages with Operators

The two schemata **F**(x_1, x_2,…, x_m) = x_{m+1} and **P**(y_1, y_2,…, y_m) define the class of atomic formulas in a complete way only if the scopes of the meta-variables x_1, x_2,…, x_m, x_{m+1} and y_1, y_2,…, y_m appearing in them are defined in the right way. Of course they may represent either individual names or object variables of the right semantic category. Note, however, that if **F**(x_1, x_2,…, x_m) = x_{m+1} is a wff then **F**(x_1, x_2,…, x_m) is an expression that represents the same objects as those represented by x_{m+1}. Thus besides individual names and object variables these expressions should be classified as ***nominal formulas***, more precisely ***compound*** ones. Moreover, we may accept the following convention.

6.7.1. SEMANTIC CATEGORIES OF NOMINAL EXPRESSIONS: Suppose $\mathbf{F}(x_1, x_2, \ldots, x_m) = x_{m+1}$ is a wff. Then $\mathbf{F}(x_1, x_2, \ldots, x_m)$ is of the same semantic category as that of x_{m+1}

In virtue of the principle 5.6.2 and convention 6.7.1 the class of atomic wffs is larger than that determined by schemata $\mathbf{F}(x_1, x_2, \ldots, x_m) = x_{m+1}$ and $\mathbf{P}(y_1, y_2, \ldots, y_m)$ under the interpretation that restrict the scopes of meta-variables $x_1, x_2, \ldots, x_m, x_{m+1}$ and $y_1, y_2, \ldots, y_m$ to non-compound ones. The following definition summarizes the above discussion.

6.7.2. SCHEMATA OF ATOMIC WFFS. Suppose $\mathbf{F}$ denotes a m-ary function $\mathbf{P}$ denotes a n-ary relation. Then the two schemata $\mathbf{F}(x_1, x_2, \ldots, x_m) = x_{m+1}$ and $\mathbf{P}(y_1, y_2, \ldots, y_n)$ represent all atomic wff's that can be formed by means of these two terms, provided that every meta-variables appearing in those schemata represents all nominal expressions of the semantic category characteristic of the argument that the meta-variable represents.

6.8. First Order Languages

A first order language was (cf. XXX) tentatively defined as a language whose atomic sentences are formed by predicates and operators and whose compound sentences are formed by classical logical constants. The definitions we have provided in this and the previous section explicate both the concept of an atomic formula (wff) and that of a compound formula of the mentioned kinds. Thus the concept of a ***first order language*** has been rigorously defined.

There is one more concept that was introduced in a preliminary way and can be rigorously defined now, Recall that a special category of compound expressions of first order languages are nominal formulas. Recall also (cf. XXX) that we have divided individual names into two kinds: proper names and ***descriptions***. The latter all nominal formula that do not variables.

Examples. **`born-in>`**`(Hašek)` (the place where Hašek was born) or **`father>`**`(John)` (the father of John).

7. Interpretations

7.1. Syntax vs. Semantics

The two previous chapters were not exclusively devoted to syntactical matters for we were discussing the semantic role of descriptive terms. In particular we were pretending that the terms that we are dealing with apply to objects forming three universes **H**, **T** and **L**. In fact, however, the question of what are those universes was of little relevance, if any, for the grammatical issues we were discussing. We might as well treat them as tree abstract sets composed of arbitrary objects. Syntactical analyses of predicate languages (notably first order predicate languages) can be carried out without undertaking any semantic issue.

There are two rather radically different ways in which semantic problems can be approached. One involves the idea of "intended interpretation." As a rule, in our semantic comments offered on various occasions we tacitly applied that approach. It consists in pretending that given a discourse, the meaning of its descriptive terms is fixed. Thus if a discourse concerns the objects in the universes **H**, **T** and **L** we know exactly what these objects are. When one of predicates of the discourse is **`mother`** we know what is the extension [**`mother`**] of that predicate (we know a procedure that allows us to tell couples (x,y) to which this predicate refers from those to which it does not), etc.

The problem with the idea of intended interpretation is that it is not a logical concept. The logician is not the right person to decide whether $(x,y) \in \mathbf{H} \times \mathbf{H}$ and whether $(x,y) \in$ [**`mother`**]. The substantial part of logical analyses should be thus carried out (the second approach) without appealing to the idea of intended interpretation and using the concept of an (arbitrary) interpretation instead.

7.2. Interpretations

In the most general sense of the word an interpretation is an assignment of meanings to words. In the case of first order predicate language by an ***interpretation*** we shall mean an arbitrary assignment of "extensions" to descriptive terms. Note that in this definition the term "extension" gains a new meaning. Using it we do not mean any longer by the extension of a term the extension that the term has under the intended meaning (interpretation).

The following notation should help clarifying the matter. Suppose $\mathfrak{I}$ is the interpretation we are using in our semantic consideration of the same predicate language we have been discussing on various occasions in two previous chapters. Then the set of objects (they need not be persons any longer) that under the assignment $\mathfrak{I}$ are meant to be in **H**, call them "$\mathfrak{I}$-people", will be denoted by $\mathbf{H}_{\mathfrak{I}}$, the set of "$\mathfrak{I}$-events" will be denoted by $\mathbf{E}_{\mathfrak{I}}$ and finally the set of "$\mathfrak{I}$-locations" will be denoted by $\mathbf{L}_{\mathfrak{I}}$.

Once "$\mathfrak{I}$-universes" are defined, one may extend the interpretation on the descriptive terms in such a way that the semantic categories of interpreted expressions are respected. Thus

> **7.2.1. THE PRINCIPLE OF SEMANTIC COHERENCE OF AN INTERPRETATION**. If the semantic category of an individual name p is that of a name of an object in a universe **U**, the extension $[p]_{\mathfrak{I}}$ of p under $\mathfrak{I}$ should be an element of $\mathbf{U}_{\mathfrak{I}}$. If the semantic category of a predicate **P** is that of symbol of a relation whose elements are in $\mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n$ then the extension $[\mathbf{P}]_{\mathfrak{I}}$ of that predicate under $\mathfrak{I}$ should be a relation whose elements are in $(\mathbf{U}_1)_{\mathfrak{I}} \times (\mathbf{U}_2)_{\mathfrak{I}} \times \ldots \times (\mathbf{U}_n)_{\mathfrak{I}}$. Finally, if the semantic category of an operator **F** is that of symbol of a function from $\mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n$ into $\mathbf{U}_{n+1}$ then the extension $[\mathbf{F}]_{\mathfrak{I}}$ of that operator under $\mathfrak{I}$ should be a function from $(\mathbf{U}_1)_{\mathfrak{I}} \times (\mathbf{U}_2)_{\mathfrak{I}} \times \ldots \times (\mathbf{U}_n)_{\mathfrak{I}}$ into $(\mathbf{U}_{n+1})_{\mathfrak{I}}$.

7.3. The logical conception of truth

Call a first order language *L* combined with an arbitrary interpretation $\mathfrak{I}$ a ***first order interpreted language***. Given a sentence α of *L* and an interpretation $\mathfrak{I}$ for that language, we will write $[\alpha]_{\mathfrak{I}} = 1$ if α is true under the interpretation $\mathfrak{I}$ and we will write $[\alpha]_{\mathfrak{I}} = 0$ if it is false.

> **7.3.1. TRUTH CONDITIONS FOR ATOMIC SENTENCES**:
>
> (i) $[\mathbf{P}(p_1,\ldots,p_1)]_{\mathfrak{I}} = 1$ iff $([p_1]_{\mathfrak{I}},\ldots,[p_1]_{\mathfrak{I}}) \in [\mathbf{P}]_{\mathfrak{I}}$.
>
> (ii) $[p = q]_{\mathfrak{I}} = 1$ iff $[p]_{\mathfrak{I}} = [q]_{\mathfrak{I}}$

7.4. Intended vs. Admissible Interpretations

To say that a sentence α of a first order language is true without indicating explicitly the relevant interpretation is to say that α it is true relative to the "intended interpretation", cf. 6.5. There are two things that should be noticed on this occasion.

Firstly, as we have noticed already (cf. XXX) the meaning of an informal expression may change from one discourse to another). Secondly, hardly ever the intended interpretation characteristic of a specific discourse is defined in a unique way. Rather it is defined "partially", i.e. by means of a set of various conditions that rule out some interpretations as "incorrect" or "unacceptable". And if so, the informal meaning of descriptive terms is determined by the set of acceptable interpretations (cf. XXX) than a single intended interpretation.

7.5. Non-Extensional Predicates

The following is a paradigmatic example that demonstrates that truth conditions 7.3.1 are not be as obvious as one might believe them to be. Formalize *John knows that the capital of Lithuania is Vilnus* as

$$^{1}\mathbf{knows\text{-}that}^{2}\mathbf{\text{-}is}^{3}(\mathtt{John}, \mathbf{capital\text{-}of}(\mathtt{Lithuania}), \mathtt{Vilnus})$$

Suppose John's geographical knowledge is poor and the analyzed sentence is (under the intended interpretation) false. On the other hand, John surely knows that Vilnius is Vilnius and hence

$$^{1}\mathbf{knows\text{-}that}^{2}\mathbf{\text{-}is}^{3}(\mathtt{John}, \mathtt{Vilnius}, \mathtt{Vilnius})$$

is true. If 7.3.1 applied to the discussed sentences that could not happened. Indeed, under the intended interpretation [**capital-of**(Lithuania)] = [Vilnus] and hence the two sentences cannot have different truth-values.

The conclusion that follows from the above discussion is that conditions 7.3.1 are not generally valid. We define a class of languages in which they are valid in the next Section.

7.6. Extensional Languages

Suppose for some nominal expressions p and q, and an interpretation $\Im$, $[\mathtt{p} = \mathtt{q}]_{\Im} = 1$. Now consider any two wff's α, β such that they become identical if all their parts that are either of the form p or of the form q are replaced by one of these two expressions, say p. Write $\alpha \approx_{\mathtt{p//q}} \beta$ in order to state that α and β are related in the way described.

> **7.3.1. EXTENSIONAL PREDICATE LANGUAGES**: A predicate language L is ***extensional*** (in the weak sense of the word, cf. XXX) iff the following

condition is satisfied. Suppose $[\mathtt{p} = \mathtt{q}]_{\mathfrak{I}} = 1$. Then for any two wff's α, β such that $\alpha \approx_{\mathtt{p//q}} \beta$, $[\alpha]_{\mathfrak{I}} = [\beta]_{\mathfrak{I}}$.

8. Semantics of truth-functional connectives

8.1. Truth-postulates for semantic connectives

The following postulates (stated with the help of the concept of truth and hence called ***truth-postulates***) define the meanings of classical connectives.

8.1.1. TRUTH-POSTULATES FOR THE CLASSICAL CONNECTIVES:

$[\neg]$	$[\neg\alpha]_{\Im} = 1$ iff $[\alpha]_{\Im} = 0$
$[\wedge]$	$[\alpha \wedge \beta]_{\Im} = 1$ iff $[\alpha]_{\Im} = [\beta]_{\Im} = 1$
$[\vee]$	$[\alpha \vee \beta]_{\Im} = 0$ iff $[\alpha]_{\Im} = [\beta]_{\Im} = 0$
$[\rightarrow]$	$[\alpha \rightarrow \beta]_{\Im} = 1$ iff $[\alpha]_{\Im} \leq [\beta]_{\Im}$
$[\equiv]$	$[\alpha \equiv \beta]_{\Im} = 1$ iff $[\alpha]_{\Im} = [\beta]_{\Im}$

The following "truth-table" provides and alternative way for characterizing the meanings of the above connectives:

α	β	$\neg\alpha$	$\alpha \wedge \beta$	$\alpha \vee \beta$	$\alpha \rightarrow \beta$	$\alpha \equiv \beta$
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

One might wonder if the accepted truth-postulates for classical connectives are consistent with the common-sense meaning of their informal counterparts. The answer is: not in every case. We are going to discuss this point in the next chapter.

8.2. Mutual Definability of Classical Connectives

An inspection of postulates 8.1.1 (or the corresponding truth table) shows that some of classical connectives are definable in terms of others. Thus e.g. every two connectives of which one is negation suffice for defining all the remaining ones.

Example. For all α and β and all interpretations $\Im$ the truth vales of conditionals $\alpha \rightarrow \beta$ is exactly the same as that of formulas of the form $\neg(\alpha \wedge \neg\beta)$. Thus we may accept $\wedge$ and $\neg$ as primitive terms and introduce implication connective $\rightarrow$ by postulating $(\alpha \rightarrow \beta) =_{df} \neg(\alpha \wedge \neg\beta)$.

8.3. Tautologies

A sentence that is true under every interpretation is called ***logically true*** or alternatively it is called a ***tautology***. We shall state that α is a tautology by writing $\models \alpha$.

The following are selected of schemata of tautologies:

Law of the Excluded Middle	$\models \alpha \vee \neg\alpha$
Law of Contradiction	$\models \neg(\alpha \wedge \neg\alpha)$
Law of Double Negation	$\models \alpha \equiv \neg\neg\alpha$

In order to prove that e.g. $\alpha \vee \neg\alpha$ is a schema of tautologies we argue as follows. Whichever might be the sentence α, and whichever might be the interpretation $\mathfrak{I}$, there are at most two possibilities: either $[\alpha]_{\mathfrak{I}} = 1$ or $[\alpha]_{\mathfrak{I}} = 0$. Accordingly $[\neg\alpha]_{\mathfrak{I}} = 0$ or $[\neg\alpha]_{\mathfrak{I}} = 1$. But in both cases $[\alpha \vee \neg\alpha]_{\mathfrak{I}} = 1$.

The proof that a schema of sentences that involves two meta-variables representing sentences, say α and β is a schema of tautologies requires discussing four possibilities. An example:

Duns Scotus' Law $\models \alpha \to (\neg\alpha \to \beta)$

The proof of validity of Duns Scotus' law might be presented in the form of the following truth table.

α	β	$\neg\alpha$	$\neg\alpha \to \beta$	$\alpha \to (\neg\alpha \to \beta)$
1	1	0	1	1
1	0	0	1	1
0	1	1	1	1
0	0	1	0	1

8.4. Two Concepts of Logical Consequence for First Order Languages

This is not the notion of logical tautology but that of logical consequence that is of chief interest of logic. However, one way to define the latter is the following:

8.4.1. THE IMPLICATIONAL CONCEPT OF LOGICAL CONSEQUENCE: *A sentence α is a* ***logically consequence*** *from sentences* $\beta_1, \beta_2, \ldots, \beta_n$ *iff* $\models (\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_n) \to \alpha$.

An alternative way is the following:

8.4.2. THE SEMANTIC CONCEPT OF LOGICAL CONSEQUENCE: *A sentence* α *is a* ***logically consequence*** *from sentences in X, in symbols* $X \models \alpha$ *iff there is no interpretation* $\Im$ *such that for all* $\beta \in X$, $[\beta]_\Im = 1$ *and* $[\alpha]_\Im = 0$.

The two concepts of entailment are related one to another by the following

8.4.3. DEDUCTION THEOREM: *For all sentences* $\beta_1, \beta_2, \ldots, \beta_n, \alpha$ *the following two conditions are equivalent:*

(i) $\models (\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_n) \rightarrow \alpha$,
(ii) $\beta_1, \beta_2, \ldots, \beta_n \models \alpha$

8.5. Laws of Logical Truth vs. Deduction Laws

If sentences of a kind (typically: sentences defined by a schema) are logically true, then a statement to the effect that it is so will be referred to as a ***law of logical truth***. Thus e.g. $\models \alpha \vee \neg\alpha$ (i.e. Law of Excluded Middle, cf. 8.2) is a law of logical truth. Now, if a sentence logically follows from sentences to which it is related in a specific way (typically: the sentences in question follow under specific schemata), then a statement to the effect that it is so will be called a ***deduction law***.

An example (a rather special one) of a deduction law is $\alpha \models \alpha$, i.e. the statement to the effect that every sentence α is a logical consequence of itself. For more examples of deduction laws that do not involve any logical constants (we shall call them ***universal***) see XXX. Amongst the deduction laws that involve logical constants, of special significance is ***Modus Ponens***: Suppose α, β are sentences, then

$$\alpha, \alpha \rightarrow \beta \models \beta.$$

Besides deduction laws of the kind exemplified by Modus Ponens there are ***conditional*** deduction laws. They will be presented as "fractions" to be read: if the "numerator schemata" are valid so is the "dominator schema". The following is the conditional variant of Modus Ponens:

$$\frac{X \models \alpha,\ Y \models \alpha \rightarrow \beta}{X \cup Y \models \beta}$$

(In order to derive unconditional Modus Ponens from its conditional variant, put in the latter $X = \{\alpha\}$, $Y = \{\alpha \to \beta\}$. Then in virtue of Identity Law, $\alpha \models \alpha$, the conditions $X \models \alpha$, $Y \models \alpha \to \beta$ are satisfied and hence also $X \cup Y \models \beta$, i.e. $\alpha, \alpha \to \beta \models \beta$ must be true).

8.6. Universal Deduction Laws

The universal deduction laws we are going to state take easier to grasp form if the notation $X \models \alpha$ is replaced by $\alpha \in \mathrm{Cn}(X)$.

8.6.1. Suppose X and Y are sets of sentences. Then the following obtains:

***Triviality Law*: $X \subseteq$ Cn(X)**
Montonicity Law: If $X \subseteq Y$, then $\mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y)$
Closure Law: $\mathrm{Cn}(\mathrm{Cn}(X)) = \mathrm{Cn}(Y)$

9. Logic of Knowledge vs. Logic of Conversation

9.1. Discourses and Belief Systems

Semantic analysis based of the truth-postulates for logical constants ignores pragmatic aspects of communication. In order to interpret correctly what one is saying (one's ***utterance***) the other participants of the discourse (conversation) must have a good idea of both one's ***background knowledge*** (the beliefs that one considers to be conclusively justified) and the goal one seeks to achieve engaging in the discourse. A ***cooperative discourse*** is a discourse whose participants seek to achieve the same goal and do not behave counterproductively. For instance, they do not hide information relevant to the goal of the discourse. Thus e.g., if being a party of such a discourse, somebody is saying *John has two children*, the other participants are entitled to conclude that *John has exactly two children* (no information must be hidden). Such a conclusion that goes beyond what is available by purely logical analysis of the utterance (as well as the reasoning by means of which one arrives at it) is called ***implicature***. The rules of arriving at implicatures were stated by Grice.

9.2. Rules of "Conversation"

Changing the term 'conversation' to discourse and assuming the cooperative nature of the discourse we can present Grice's rules as follows:

> **Principle of Cooperation.** *Every participant should make such input into the exchange of views as is expected at the given stage in the discourse from the point of view of the aims of the discourse.*
>
> **Quality maxim.** *Discourse participants should not put forward views of whose falsehood they are sure of or even views which aren't adequately justified.*
>
> **Quantity maxim.** *No more or less information should be given than is necessary at the given stage of the exchange of views.*
>
> **Relevance maxim.** *Propositions which are not significant to the discourse should not be stated.*
>
> **Method maxim.** *The method in which the participants formulate their statements should be as communicative, as concise, as free of vagueness and ambiguity as possible.*

9.3. Paradoxes of material implication

Consider the following two sentences:

(1) *If Warsaw is in Poland, then whales are mammals.*

(2) *If Warsaw has about three millions inhabitants, then it has about one million*.

Both sentences are true if *if…then* is interpreted as two-value implication $\rightarrow$. On the other hand whether whales are mammals has nothing to do with where Warsaw is, and thus (1) states, though implicitly, something that is not true. To maintain that (2) is true seems to be just an affront to sound reason. It cannot be that if Warsaw has three million participants it has one million. Examples of less or more striking divergence between the two-valued and common-sense meaning of *if…then* are called ***paradoxes of material implication*** (here "material" means "two-valued")

Counter to the once-dominant view, paradoxes of implication are not a semantic phenomenon but, rather, a pragmatic phenomenon. The feeling of the unsuitableness of statements like (1) and (2) is not caused by violation of the principles which determine the truth-value of a proposition. Rather, it arises due to ignoring the principles of linguistic communication such as the Gricean maxims. The key commandment of pragmatics is not 'be truthful' but 'be trustful'. Suppose in reply to the question asked at 10 a.m. "what the next train to Leipzig is?" Paul hears" "There is a train going to Leipzig at 4 p.m." If there is such a train, the answer is true. However, Paul has every right to claim that he was deceived by the informer, if there is an earlier train.

9.4. Proofs by Reduction ad Absurdum

Let us return to (2) and the problem of its apparent self-contradiction. It will be more convenient if, instead of (2) we discuss a mathematical equivalent:

(3) *If the set* $\{\varnothing\}$ *has 10 elements, then it has precisely one.*

Conditionals with a false antecedent are treated in mathematics as counterfactual and counted as true. There is something that very powerfully motivates this convention. Assume that we want to show that a certain proposition α is true in the sense that it follows from postulates considered to be true. A typical way to meet this task is the following. Add $\neg\alpha$ to the postulates and try to show that thus enriched set of premises (postulates along with $\neg\alpha$) logically entails α. If so, provided that the postulates are true, the sentence $\neg\alpha$ cannot be true. For if it were, one could not be able to demonstrate that α is a logical consequence of thus selected premises; a logical consequence of true premises must be true; under the standard meaning of negation both $\neg\alpha$ and α cannot be true. This way of reasoning is called *reductio to absurdum*.

If under the adopted convention, all conditionals whose antecedent denies precedent were false, proving that $\neg\alpha \rightarrow \alpha$ would be impossible; a sound proof cannot result in proving a false statement. And if so, it would be impossible to use *reduction ad absurdum* as a method of proving mathematical theorems.

An analysis of mathematical arguments shows that the meaning of *if then* in mathematical arguments conform to the meaning of two valued $\rightarrow$. But this does not imply that the two coincide on ever occasion. Using a conditional *if* α *then* β in a discourse may yield a false implicture. If this is a case the statement *if* α *then* β one might be reluctant to treat the conditional *if* α *then* β as true.

9.5. Non-Classical Implications

Before Grice formulated his maxims treating paradoxes of material implication as a pragmatic phenomenon (true implication might yield false implicature) was not possible. The typical way of approaching this problem consisted in criticizing the adequacy of the truth-functional analysis of implication. Thus e.g. C. I. Lewis advocated the view that the common-sense meaning of *if* α *then* β is better rendered by the $\alpha \Rightarrow \beta$ called strong implication rather than by $\alpha \rightarrow \beta$. Under his analysis ***strong implication*** is to be understood as follows:

$$\alpha \Rightarrow \beta \equiv_{df} \textit{it is impossible that } \beta \textit{ is false and } \alpha \textit{ is true}.$$

One might rewrite this as follows $\alpha \Rightarrow \beta \equiv_{df} \neg\Diamond(\alpha \wedge \neg\beta)$, with $\Diamond$ standing for *it possible that* and all the remaining connectives being two-value. Use $\Box$ to stand for *it necessary that* meant to be the same as $\neg\Diamond\neg$. Then $\neg\Diamond(\alpha \wedge \neg\beta)$ might be successively rewritten as (1) $\neg\Diamond\neg\neg(\alpha \wedge \neg\beta)$, (2) $\Box\neg(\alpha \wedge \neg\beta)$, (3) $\Box(\alpha \rightarrow \beta)$. Thus strong implication is definable in terms of two-value implication and necessity connective.

Modal logic – such is the name of the logic which studies the formal properties of the connectives *it is possible that* and *it is necessary that* – is based upon the idea of possible words. The sentence $\Box\alpha$ is defined as true is it is true in all the possible worlds or, sometimes, in all the "accessible" possible worlds. The concept of an "accessible world" requires a separate definition, of course.

One of the developments which were to result in the replacement of "faulty" classical logic was based upon the proposal that the truth-value of *if* α *then* β statements can not be decided just by the truth-values of the antecedent and the consequent. It was argued that the existence of a connection between the content of α and β was a necessary condition for the truth of the conditional. The paradoxical nature of the whales statement (see (1) in 9.3) was due to the lack of just such a connection. This direction of investigation was begun by the work of A. R. Anderson and N. D. Belnap Jr. and led to the creation of a whole class of ***relevance logics***.

9.6. The Common-Sense vs. The Formalized Meaning of Connectives

Implication is not the only connective whose informal meaning does not coincide with the "truth-table" meaning of its formalized counterpart. For instance, to say *Peter visited Ann and changed his plans* and to say *Peter changed his plans and visited Ann* is two say two different things.

10. Predicate Calculus

10.1. The Common-Sense Meaning of Generalizations

It is fairly clear how sentences of the form $\forall x(\phi(x))$ or $\exists x(\phi(x))$ (both will be called ***generalizations***) should be understood. By saying that $\forall x(\phi(x))$ one says that the condition $\phi(x)$ is true of all objects represented by the variable x (e.g of all people if x represents people). By saying that $\exists x(\phi(x))$ one says that $\phi(x)$ is true of at least one such object. The problem we are going to discuss will not be how the quantified sentences should be understood but rather how the commonsense idea of their truth should be expressed in precise terms.

To begin with note the following. To say that the condition $\phi(x)$ is true of an object $\boldsymbol{a}$ selected from all the objects represented by the variable x is to say that if a is a proper name for $\boldsymbol{a}$ then the sentence $\phi(a)$ is true. Unfortunately the object $\boldsymbol{a}$ may have no proper name in the analyzed language. For instance, in no language we use, we do not have names for everybody.

A sentence that results from $\phi(x)$ by replacing x by a proper name of the same semantic category as that of x is called an ***instance*** of $\phi(x)$. Although we cannot express the common-sense idea of truth of $\forall x(\phi(x))$ (of $\exists x(\phi(x))$) by saying that all (some) instances of $\phi(x)$ are true, we certainly may express it by saying that all (some) "possible instances" (available by adding new proper names) of $\phi(x)$ are true. The postulate of truth-functionality of quantified sentences should thus be spelled out as one that requires the truth-values of $\forall x(\phi(x))$ and $\exists x(\phi(x))$ to be determined by the truth values of "possible instances" of $\phi(x)$.

10.2. Valuations

Suppose $\boldsymbol{a}$ is an object represented by the variable x under an interpretation $\mathfrak{I}$. Then (unless this violates other stipulations one has already accepted), one may use x as a proper name for $\boldsymbol{a}$. The fact that, on a given occasion, a variable x appearing in a wff $\phi(x)$ as a free variable should be treated as a proper name for $\boldsymbol{a}$ must be somehow stated explicitly. The practical aspects of the relevant procedure will not be of interest for us. It suffice that we agree that whenever free variables $x_1,\ldots,x_n$ appearing in a wff $\psi(x_1,\ldots,x_n)$ are treated as proper names for $\boldsymbol{a}_1,\ldots, \boldsymbol{a}_n$ respectively, we shall notify this fact by writing

$$\psi(x_1,\ldots,x_n)[x_1/\boldsymbol{a}_1,\ldots,x_n/\boldsymbol{a}_n]$$

The procedure of assigning individual objects to free variables is called ***valuation***.

10.3. Truth-postulates for quantified sentences

The meaning of classical quantifiers is defined by the following

10.3.1. TRUTH POSTULATES FOR QUENTIFIERS:

($\forall$) $[\forall x(\phi(x))]_{\Im} = 1$, i.e. the sentence $\forall x(\phi(x))$ is true under interpretation $\Im$, if and only if $[\phi(x)[x/\boldsymbol{a}]]_{\Im} = 1$, for every object ***a*** represented by variable x under interpretation $\Im$.

($\exists$) $[\exists x(\phi(x))]_{\Im} = 1$, i.e. the sentence $\exists x(\phi(x))$ is true under interpretation $\Im$, if and only if $[\phi(x)[x/\boldsymbol{a}]]_{\Im} = 1$, for an object ***a*** represented by variable x under interpretation.

10.4. An example

Let us examine the truth-value of the sentence

(1) $\forall a(\mathbf{person}(a) \rightarrow \exists b(\mathbf{person}(b) \wedge \mathbf{mother}(a,b))$

by appealing to postulates ($\forall$) and ($\exists$). Select an interpretation $\Im$. By ($\forall$), in order for (1) to be true under $\Im$ we must have

(2) $[(\mathbf{person}(a) \rightarrow \exists b(\mathbf{person}(b) \wedge \mathbf{mother}(a,b))[a/\mathbf{p}]]_{\Im} = 1$

for all **p** in the scope that $\Im$ assigns to a. In virtue of the truth-postulate for $\rightarrow$ (if one decided to argue on the base of the commonsense meaning of *if...then* the conclusion would be the same) (2) is false if

(3) $[\mathbf{person}(a)[a/\mathbf{p}]]_{\Im} = 1$, i.e. $\mathbf{p} \in [\mathbf{person}]_{\Im}$
(4) $[(\exists b(\mathbf{person}(b) \wedge \mathbf{mother}(a,b))[a/\mathbf{p}]]_{\Im} = 0$

Under postulate ($\exists$) in order for (4) to be false, there must not exist any element **q**, in the scope of that b has under interpretation $\Im$, such that

(5) $[(\mathbf{person}(b) \wedge \mathbf{mother}(b, a))[a/\mathbf{p}, b/\mathbf{q}]]_{\Im} = 1$

By appealing to either the truth-postulate for $\wedge$ or the commonsense meaning of *and* we conclude that (5) obtains iff both

(6) $[\mathbf{person}(b)[b/\mathbf{q}]]_{\mathfrak{I}} = 1$, i.e. $\mathbf{q} \in [\mathbf{person}]_{\mathfrak{I}}$
and
(7) $[\mathbf{mother}(b, a)[a/\mathbf{p}, b/\mathbf{q}]]_{\mathfrak{I}} = 1$, i.e. $(\mathbf{p}, \mathbf{q}) \in [\mathbf{mother}]_{\mathfrak{I}}$

This ends the logical part of analysis. In order to learn whether for given **p** and **q**, $\mathbf{p} \in [\mathbf{person}]_{\mathfrak{I}}$, $\mathbf{q} \in [\mathbf{person}]_{\mathfrak{I}}$, and $(\mathbf{p}, \mathbf{q}) \in [\mathbf{mother}]_{\mathfrak{I}}$ and thus eventually find out whether (1) is true or not one has to resort to the definition of $\mathfrak{I}$. Of course, one should resort to the intuitive meanings of the relevant expressions if $\mathfrak{I}$ is meant to be the intended interpretation.

10.5. Truth-functionality

Suppose that for an interpretation $\mathfrak{I}$ two sentences α and β are equivalent, i.e. $[\alpha \equiv \beta]_{\mathfrak{I}} = 1$. Does this imply that these two sentences are "exchangeable *salva veriate*" in any sentential context in which they appear? In other word does this imply that given any couple of sentences φ and φ′ such that $\varphi \approx_{\alpha//\beta} \varphi'$, $[\varphi \equiv \varphi']_{\mathfrak{I}} = 1$? The answer to this question depends on the vocabulary of the language examined.

Suppose that one of the connectives with the help of which composed formulas are formed is *because*. Suppose there was a power brake that stopped a train. Suppose moreover that there were more than 200 passengers in that train. In the true sentence that states that power brake was cause of the delay replace *there was a power brake* by *there were more than 200 passengers in that train*. The replacement does not preserve the truth value. The diagnosis is the connective *because* is not truth functional connective – its meaning cannot be adequately defined in terms of truth values of the sentences to which it is applied.

The idea of truth-functionality extends on language as a whole as follows

> **10.5.1. TRUTH-FUNCTIONAL PREDICATE LANGUAGES**. A predicate language L is ***truth-fuctional*** iff the following condition is satisfied. Suppose $[\alpha \equiv \beta]_{\mathfrak{I}} = 1$. Then for any two sentences φ and φ′ such that $\varphi \approx_{\alpha//\beta} \varphi'$ $[[\varphi \equiv \varphi']_{\mathfrak{I}} = 1$.

A predicate language is said to be ***extensional*** (in the strong sense of the word) if it is both weakly extensional (cf. XXX) and truth-functional.

10.6. Logical Consequences of wffs

The two chief tasks of logic are the following. Firstly, logic should answer the question what it means for a sentence α to be logically entailed by (to be logically implied by, to logically follow from, to be a logical consequence of) a set of sentences X, in symbols $X \models \alpha$. Secondly, logic should offer tools for proving that $X \models \alpha$ whenever the entailment take place. Seemingly the rules for forming the proof should be implicit in the clauses of the definition of logical consequence. They are, but the relationship between entailment and provability is fairly complex. In this chapter we are going to take a closer look at the matter.

Though the object of our prime concern remains sentences, we shall expand the definition 8.3.2 of logical consequence on wffs.

Recall that an interpretation $\Im$ for the examined language L defines the scopes of the variables of that language. Let v assigns to every variable x of L an object $v(x)$ within the scope of the variable x defined by $\Im$.

Now, let α be a wff with $x_1,\ldots,x_k$ being its only free variables. Let v be a referent assignment. Then we define:

$$\alpha[v] = \alpha[x_1/v(x_1),\ldots,x_k/v(x_k)]$$

Moreover, given a set of wffs X, we define $X[v]$ to be the set of all formulas of the form $\alpha[v]$ such that α is in X.

> **10.5.1. LOGICAL CONSEQUENCE DEFINED FOR WFFS**: α is a ***logical consequence*** of wffs in X, in symbols $X \models \alpha$, iff for every interpretation $\Im$, and every valuation v to variables elements of their scopes defined by $\Im$, if $\Im(X[v]) = 1$ then $\Im(\alpha[v]) = 1$.

11. Laws of Natural Deduction

11.1. The Laws of Natural Deduction for Identity

The term “natural deduction” is related to Gentzen’s discovery (in fact Gentzen’s and Jaskowski’s) discovery that formal proofs can be presented in the form of a sequence of “steps”, each step executed in accordance with a “natural” (conforming to the way in which the mathematicians argue) way of carrying out proofs.

To begin with let us state laws of natural deduction for identity and equivalence. In spite of the fact that these two logical constants are of entirely different “syntactic category” (their role in forming sentences is different) there is a considerable similarity between the laws that characterize them.

11. 1.1 NATURAL DEDUCTION SCHEMATA FOR IDENTITY: Let p and q be nominal expressions and let α and β be wffs. Then

Identity Law

$$\models \mathrm{p}=\mathrm{p}$$

Replacement Law: Let $\alpha \approx_{\mathrm{p}//\mathrm{q}} \beta$, then

$$\frac{X \models \mathrm{p}=\mathrm{q},\ Y \models \alpha}{X \cup Y \models \beta}$$

11.2. Laws of Natural Deduction for Equivalence

11. 1.1 NATURAL DEDUCTION SCHEMATA FOR EQUIVALENCE: Let α and β be wffs. Then

Identity Law

$$\models \alpha \equiv \alpha$$

Replacement Law: Let $\varphi \approx_{\alpha//\beta} \varphi'$, then

$$\frac{X \models \alpha \equiv \beta, \;\; Y \models \varphi}{X \cup Y \models \varphi'}$$

11.1. Laws of Natural Deduction for Connectives

Modus Ponens:

$$\alpha, \alpha \to \beta \models \beta$$

$$\frac{X \models \alpha, \; Y \models \alpha \to \beta}{X \cup Y \models \beta}$$

$\to$ - Introduction:

$$\frac{X, \alpha \models \beta}{X \models \alpha \to \beta}$$

Reduction ad absurdum:

$$\frac{X, \neg\alpha \models \alpha}{X \models \alpha}$$

Laws of overcompletion

$$\alpha, \;\; \neg\alpha \models \beta$$

$$\frac{X \models \alpha, \;\; Y \models \neg\alpha}{X \cup Y \models \beta}$$

$\wedge$-Introduction:

$$\alpha, \beta \;\; \models \alpha \wedge \beta$$

$$\frac{X \models \alpha, \;\; Y \models \beta}{X \cup Y \models \alpha \wedge \beta}$$

∧-Reduction:

$$\alpha \wedge \beta \models \alpha$$

$$\frac{X \models \alpha \wedge \beta}{X \models \alpha}$$

Alternative deduction:

$$\frac{X, \alpha \models \gamma; \; Y, \beta \models \gamma}{X \cup Y, \alpha \vee \beta \models \gamma}$$

∨-Introduction:

$$\frac{X \models \alpha}{X \models \alpha \vee \beta}$$

11.2. Schemata of natural deduction for quantifiers

Instantiation:

$$\frac{X \models \forall x \phi(x)}{X \models \phi(x)}$$

Universal generalization:

$$\frac{X \models \phi(x)}{X \models \forall x(\phi(x))} \; *)$$

*) Proviso: *The variable x may occur free in no formula in the set X.*

Designation:

$$\frac{X \models \exists x(\phi(x))}{X \models \phi(\mathrm{p})} \; *)$$

*) Proviso: p *should be a name for the object whose existence is stated by the formula* $\exists x(\phi(x))$. *If no such name is in the language examined, a name of this kind may be added to the vocabulary of the language.*

Existential generalization:

$$\frac{X \models \phi(x)}{X \models \exists x(\phi(x))}$$

11.3. An Example of a Proof of a Law of Logic

Consider the schema $\forall x(\phi(x)) \models \phi(x)$. One may show that all its instances obtain and hence it represents a deduction law.

Indeed, suppose $\Im(\forall x(\phi(x))[v]) = 1$ for an interpretation $\Im$ and a referent assignment v consistent with $\Im$. Note that for a given v, $\forall x(\phi(x))[v]$ is a sentence. If all free variables in the wff $\forall x(\phi(x))$ are $x_1,\ldots,x_k$, then $\forall x(\phi(x))[v] = \forall x(\phi(x))[x_1/v(x_1),\ldots,x_k/v(x_k)]$. By the postulate $(\forall)$ for $\forall$, $\Im(\forall x(\phi(x))[x_1/v(x_1),\ldots,x_k/v(x_k)]) = 1$ iff $\Im((\phi(x))[x_1/v(x_1),\ldots,x_k/v(x_k)][x/\boldsymbol{a}]) = 1$, for every object $\boldsymbol{a}$ represented by x under interpretation $\Im$. Thus in particular $\Im((\phi(x))[x_1/v(x_1),\ldots,x_k/v(x_k)][x/v(x)]) = 1$. But if so we have $\Im(\phi(x)[v]) = 1$, exactly as required by (D3|=) in order for $\forall$E to hold true, which concludes the argument.

12. Knowledge Systems

12.1. Two concepts of knowledge

The term “knowledge” has two meanings. In its ***epistemological*** sense, knowledge is the set of beliefs which have been properly justified and which are true. In the ***methodological*** sense, knowledge is the set of critically developed ***belief systems*** i.e. sets of propositions put forward in order to account for certain states of affairs and/or events. Some of beliefs that form a belief system might happen not to be properly justified and, certainly, not all of them need to be true. Belief systems which make up a system of knowledge may complement and inform each other as well as provide mutually contradictory answers to the same questions – as is the case with competing theories in science.

The concept of a belief system is methodological rather than logical. On the other hand any methodological analysis of a belief system requires logical tools. Such an analysis is typically carried out with two goals in mind. One is to find out whether the system is both “internally” ***consistent***, (i.e. no proposition along with its negation follows from propositions of which the system is composed) and consistent with well-established facts and reliable hypotheses not being part of the system in a direct way. Thus, for instance, an account for specific physiological phenomena should not violate the available and well-established chemical knowledge. The other chief goal of methodological analysis is to find out whether the propositions of which the system is composed are of any epistemological significance. The standard criterion of epistemological significance is falsifibility.

Both consistency and epistemological significance are concepts related to that of consequence operation. The question we are going to discuss is of whether the consequence relation mentioned can be defined in terms of logical concepts (notably that of logical consequence) examined in the previous chapters of this guide.

12.2. Consequence Operation vs. Logical Consequence Operation

As an example that illustrates the difference between consequence operation and logical consequence operation consider the following couple of sentences: **`father`**`(Paul)` = `John` and **`older`**`(John, Paul)`. Of course neither follows logically from the other. On the other hand the latter follows from the former for an obvious reason. The meaning of the operator **`father`** and the predicate **`older`** rule out that **`older`**(`John, Paul`) might happen to be false even though **`father`**(`Paul`) = `John` were true.

Let us agree to use (|= as symbol of consequence relation. This is not a standard system; one cannot find it in logical writings. The tentative idea of consequence operation is the following:

> **12.2.1. THE INTUITIVE CONCEPTION OF CONSEQUENCE RELATION**. A sentence α ***follows*** from premises (assumptions) X, in other words α is a ***consequence*** of X, in symbols X (|= α, iff in virtue of the meanings of terms that appear in all sentences in question it impossible that the premises (the sentences that form the set X) are true and nevertheless α is false.

12.3. The Analytical Conception of Consequence Operation

At the first glance the relation between consequence and logical consequence is easy to define. Although, **`older`**(`John, Paul`) does not follow logically from **`father`**(`Paul`) = `John`, it follows logically from **`father`**(`Paul`) = `John` supplemented by two additional premises: (1) **`father`**(`Paul`) = `John` → **`parent`**(`Paul, John`) and (2) $\forall x \forall y$(**`parent`**(x, y) → **`older`**(x, y)). This remark suggest that, to say that α is a ***consequence*** of X is to say that α is a logical ***consequence*** of $X \cup \wp$, where $\wp$ is the set of definitions and postulates that define the meanings of terms that appear in the relevant formulas.

Both the concept of a definition and that of a postulate is relevant concept. A sentence which is a definition (postulate) within a belief system may not be within another. Given a belief system **B** denote by $\wp_{\mathbf{B}}$ the set of all definitions and postulates characteristic of that system. For many belief systems (say mathematical theories given in the form of an axiom system) the set $\wp_{\mathbf{B}}$ is well defined. T the same time for most of informal belief systems it is not. Let us focus our attention on those for which it is. In view of the above analyses we should examine the concept of consequence as a relative concept. Our notation should be then changed accordingly. Given a belief system **B** we shall the consequence relation characteristic of it by (**B**|=.

> **12.2.1. THE ANALITCICAL CONCEPTION OF CONSEQUENCE RELATION**. Let both X and α be defined for a specific belief system **B**. Then X (**B**|= α, iff $X \cup \wp_{\mathbf{B}} \models \alpha$

The conception we have defined is called analytical because sentences that are definitions or postulates as well as all their logical consequences are customarily called ***analytical***. All the remaining ones are then called ***synthetical***.

12.4. The Pseudo-Analytical Conception of Consequence Operation

As has been argued by Quine, except belief systems that have been axiomatized, there is no good criterion for distinguishing amongst all propositions of which the system is composed those which are definitions or postulates from those which are not. Nonetheless people who are using such system may have a fairly clear daa what within such a system follows from what. Does one who wishes to formalize such a system need to start with indicating those which are to be treated as definitions or postulates and those which do not? In view of Quine's criticism one might doubt if adopting this methodology is justified. It may suffice to divide propositions into two categories: those which are considered to be firmly confirmed and thus unquestionable and the remaining ones. Of course the role of the former in the system is the same as postulates, so we might call them ***pseudo-postulates***.

Suppose given a belief system **B** the set of pseudo-postulates of that system has been somehow (always in a less or more arbitrary manner) defined. Denote it by $\Re_{\mathbf{B}}$. an alternative way of defining the concept of consequence characteristic for a system **B** is the following

> **12.4.1. THE PSEUDO-ANALITCICAL CONCEPTION OF CONSEQUENCE RELATION**. Let both X and α be defined for a specific belief system **B**. Then X (**B**$|= \alpha$, iff $X \cup \Re_{\mathbf{B}} |= \alpha$.

12.5. Incompleteness of Verbalized Knowledge

Logic in its present form concerns verbalized knowledge. But we know things we are not able to express in the form of sentences. We know (an obvious example) how our relatives and friends look like but we are not able to fully verbalize that knowledge.

One might wonder if the concept a belief system meant to be a set of propositions provides the right tool for defining what we know. Most likely it is not. Is there any logic of non-verbalized knowledge? What might be the key assumptions on which it should be based? What could be its basic concepts? As things are now, we know no good answers to these questions.