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INFERENTIAL INTENSIONALITY

Abstract. The paper is a study of properties of quasi-consequence operation which is a key notion of the so-called *inferential approach* in the theory of sentential calculi established in [5].The principal motivation behind the quasi-consequence, *q-consequence* for short, stems from the mathematical practice which treats some auxiliary assumptions as mere hypotheses rather than axioms and their further occurrence in place of conclusions may be justified or not. The main semantic feature of the q-consequence reflecting the idea is that its rules lead from the non-rejected assumptions to the accepted conclusions.

First, we focus on the syntactic features of the framework and present the q-consequence as related to the notion of proof. Such a presentation uncovers the reasons for which the adjective "inferential" is used to characterize the approach and , possibly, the term "inference operation" replaces "q-consequence". It also shows that the inferential approach is a generalisation of the Tarski setting and, therefore, it may potentially absorb several concepts from the theory of sentential calculi, cf. [10]. However, as some concrete applications show, see e.g. [4], the new approach opens perspectives for further exploration.

The main part of the paper is devoted to some notions absent in Tarski approach. We show that for a given q-consequence operation W instead of one W-equivalence established by the properties of W we may consider two congruence relations. For one of them the current name is preserved and for the other the term "W-equality" is adopted. While the two relations coincide for any W which is a consequence operation, for an arbitrary W the inferential equality and the inferential equivalence may differ. Further to this we introduce the concepts of inferential extensionality and intensionality for q-consequence operations and connectives. Some general results obtained in Section 3 sufficiently confirm the importance of these notions. To complete a view, in Section 4 we apply the new intensionality-extensionality distinction to inferential extensions of a version of the Łukasiewicz four valued modal logic.

Key notions: consequence, proof, C-equivalence, C-equality, inference, rejection, acceptance, rule of inference, Tarski, Łukasiewicz, Ł-modal system, four-valued logicTarski, Łukasiewicz, Ł-modal logic, four-valued logic

1. Generalized inference

Let $\mathcal{L} = (For, F_1, \ldots, F_m)$ be an algebra freely generated by the set of sentential variables $Var = \{p, q, r, \ldots\}$ and the operations $F_1, \ldots, F_m$ representing the sentential connectives. In what follows $\mathcal{L}$ is called a *sentential language.* In most cases, either the language of the classical sentential logic $\mathcal{L}_k = (For, \neg, \rightarrow, \vee, \wedge, \leftrightarrow)$ with negation ($\neg$), implication ($\rightarrow$), disjunction ($\vee$), conjunction ($\wedge$), and equivalence ($\leftrightarrow$), or some of its reducts is considered. We use $Hom(\mathcal{L}, \mathcal{A})$, where $\mathcal{A}$ is an algebra similar to $\mathcal{L}$ to denote the set of all homomorphisms of the language $\mathcal{L}$ in the $\mathcal{A}$. The elements of $Hom(\mathcal{L}, \mathcal{A})$ settle interpretations of formulas of $\mathcal{L}$ in $\mathcal{A}$. In the end, the elements of $End(\mathcal{L}) = Hom(\mathcal{L}, \mathcal{L})$, are *substitutions* of formulas.

The standard notion of syntactical inference, i.e. a proof, is our base. Each pair (X, α), where $X \subseteq For$ and $\alpha \in For$ is called a *sequent.* A *rule* **R** is any set of sequents, i.e. a subset of $2^{For} \times For$. The set of all rules of inference of $\mathcal{L}$ is denoted by $Rl(\mathcal{L})$.

R is *structural* if $(X, \alpha) \in R$ implies that $(eX, e\alpha) \in R$ for every $e \in End(\mathcal{L})$. If **R** is a set of rules then we say that α is **R**-*inferred* from X, $X \vdash_{\mathbf{R}} \alpha$, whenever for some ordinal number ν there is a sequence (a proof) $\{\alpha_\mu\}_{\mu \leq \nu+1}$ such that

(p$_1$) $a = \alpha_{\nu+1}$
(p$_2$) for any $\mu^{\cdot} \leq \nu + 1$: (i) $\alpha_{\mu^{\cdot}} \in X$ or
(ii) there is $Y \subseteq X \cup \{\alpha_\mu : \mu < \mu^{\cdot}\}$ and a rule $R \in \mathbf{R}$ such that $(Y, \alpha_{\mu^{\cdot}}) \in R$.

We say that $Y \subseteq For$ is *R-closed*, where $R = (X, \alpha)$, if and only if $Y \subseteq X$ implies that $\alpha \in Y$. Y is **R**-*closed* provided that it is R-closed for every $R \in$. For any set $X \subseteq For$ let $Cn_{\mathbf{R}}(X)$ be the least set of formulas of $\mathcal{L}$ containing X and closed with respect to **R**:

$$Cn_{\mathbf{R}}(X) = \cap\{Y \subseteq For : X \subseteq Y \text{ and } Y \text{ is } \mathbf{R} - closed\} .$$

$C : 2^{For} \to 2^{For}$ is a *consequence* operation on $\mathcal{L}$ and for arbitrary $X, Y \subseteq For$ it satisfies the following Tarski's conditions:

(T0) $X \subseteq C(X)$
(T1) $C(X) \subseteq C(Y)$ whenever $X \subseteq Y$
(T2) $C(C(X)) = C(X)$.

If C is a consequence operation on $\mathcal{L}$ and $\mathbf{R}$ a set of rules of inference such that $C = Cn_{\mathbf{R}}$ then $\mathbf{R}$ is called a *base* of C.

1.1. (cf. [10]). *If* $\mathbf{R}$ *is a base of a consequence* C*, then* $\alpha \in C(X)$ *if and only if* $X \vdash_{\mathbf{R}} \alpha$.

To meet the line of the generalized approach, in [6] we were forced to neglect Tarski's reflexivity postulate (T0) and to weaken the closure condition (T2). The operation satisfying new requirements was named quasi-consequence, or q-consequence, for short. Further to this, $W : 2^{For} \to 2^{For}$is a *q-consequence* operation on $\mathcal{L}$ whenever for any $X, Y \subseteq For$

(W1) $W(X) \subseteq W(Y)$ whenever $X \subseteq Y$
(W2) $W(X \cup W(X)) = W(X)$.

Notice that to get the Tarski set of postulates it suffices to add to (W1), W(2) the reflexivity condition

(W0) $X \subseteq W(X)$.

We assume that a *q-inference* (*q-proof*) from X to $\alpha, X \vdash^*_{\mathbf{R}} \alpha$, is a sequence $\{\alpha_\mu\}_{\mu \leq \nu+1}$ defined by (p_1) and (p_2) (ii) only. Thus, $X \vdash^*_{\mathbf{R}}$ differs from $\vdash_{\mathbf{R}}$ by the fact that in the former unlimited use of the rule, of $rep = \{\{\alpha\}, \alpha : \alpha \in For\}$, is not guaranteed axiomatically.

To introduce the "*q*-notion" corresponding to $Cn_{\mathbf{R}}$, we need a special kind of a relative closure: we shall say that a set of formulas $Y \subseteq For$ is $\mathbf{R}$-*closed relative to* $X \subseteq For$ if and only if for each $(Z, \alpha) \in R \in \mathbf{R}$ if $Z \subseteq X \cup Y$, then $\alpha \in Y$. For any $X \subseteq For$ we put

$$Wn_{\mathbf{R}}(X) = \cap\{Y \subseteq For : Y \text{ is } \mathbf{R} - \textit{closed relative to } X\}$$

thus defining a mapping $Wn_{\mathbf{R}} : 2^{For} \to 2^{For}$.

If $W : 2^{For} \to 2^{For}$. is an operation on $\mathcal{L}$ and $\mathbf{R}$ a set of rules such that $W = Wn_{\mathbf{R}}$, then $\mathbf{R}$ is called a *base* of W Then, W is a *consequence* operation on $\mathcal{L}$ and for arbitrary $X, Y \subseteq For$ it satisfies the following Tarski's conditions:

1.2. (cf. [5]) *$Wn_{\mathbf{R}}$ is a q-consequence operation on $\mathcal{L}$.*

Proof. If $X_1 \subseteq X_2$ then each set of formulas of $\mathcal{L}$ is closed relative to X_1 as well. Hence (W1).

Assume now that $Wn_{\mathbf{R}}$ does not satisfy (W2) for a certain $X \subseteq For$. Then, there is a $\mathbf{R}$-closed relative to X set of formulas Y_0 which is not $\mathbf{R}$-closed relative to $Wn_{\mathbf{R}}(X) \cup X$. Let then $(Z, \alpha) \in R \in \mathbf{R}$ be a sequent such that

$$Z \subseteq Wn_{\mathbf{R}}(X) \cup X \cup Y_0 \text{ and } \alpha \notin Y_0.$$

Then $Z \not\subseteq X \cup Y_0$ (since otherwise $\alpha \in Y_0$) and therefore there must exist a $\beta \in Z \cap Wn_{\mathbf{R}}(X)$ such that $\beta \notin X \cup Y_0$. A contradiction: if $\beta \in Qn_{\mathbf{R}}(X)$, then $\beta \in Y_0 \subseteq X \cup Y_0$.

Following the construction from [10] used for proving Lemma 1.1, in a careful way, leads to

1.3. *If $\mathbf{R}$ is a base of a q-consequence W, i.e. if $W = Wn_{\mathbf{R}}$, then $\alpha \in W(X)$ if and only if $X \vdash^{*}_{\mathbf{R}} \alpha$.*

The main feature of the q-consequence W is that the *repetition rule*:

$rep = (\{\alpha\}, \alpha : \alpha \in For)$,

in general is not a rule of W. And, if, $rep \in \mathbf{R}$, then $Wn_{\mathbf{R}}$ is a consequence operation and $Wn_{\mathbf{R}} = Cna_{\mathbf{R}}$. Then for any set of formulas $X \subseteq For$: $Cn_{\mathbf{R}}(X) = X \cup Wn_{\mathbf{R}}(X)$ and $Cn_{\mathbf{R}}$ is the least consequence operation C stronger than $Wn_{\mathbf{R}}$, i.e. such that $Wn_{\mathbf{R}}(X) \subseteq C(X)$, any $X \subseteq For$. It is also worth noting that even if $rep \notin \mathbf{R}$ some formulas from X may appear in $Wn_{\mathbf{R}}(X)$. This occurs when some sequents of the rule repetition are derivable from other rules in $\mathbf{R}$.

1.4. *rep* is the only rule of inference whose presence in the Tarski's consequence paradigm is warranted by a methodological postulate (T0).

2. Inferential extensionality and intensionality

The *repetition rule* need not be unrestrainedly valid for arbitratry q-consequence W. So, in general, for some formulas α, β the condition

(1) $\alpha \in W(X)$ if and only if $\beta \in W(X)$

does not imply

(2) $W(X, \alpha) = W(X, \beta)$.

The property just mentioned allows us to distinguish two relations between formulas, and, ultimately, the inferential notions of extensionality and intensionality. The germ of the idea is the Suszko's distinction between the denotation of a sentence and its logical value, cf. [9]. In conformity with the theory of logical calculi, cf. [10], there are good reasons for introducing the concepts of "W-equivalence" and "W-identity". The first concept will be based on the idea that the logical truth of a sentence α with respect to a given set of premises X coincides with the truth value of the formula $\alpha \in W(X)$. The W-identity, on its turn, depends on the denotation and is settled by the indiscernibility of sentences in theories containing X.

Now, we shall put it in general terms. Hereafter, the symbols like $\varphi(\alpha/p)$ and $\varphi(\beta/p)$, where α, β and φ are formulas and p is a propositional variable, stand for the formulas resulting form φ by substituting the formula α(or β) for all occurrences of p. Given a q-consequence W on $\mathcal{L}$, we define two binary relations: $=_W$ and $\approx_W$ on For, putting

(*) $\alpha =_W \beta$ if and only if $W(X, \varphi(\alpha/p)) = W(X, \varphi(\beta/p))$
for every $\alpha, \beta, \varphi \in For, X \subseteq For$ and $p \in Var$.

(**) $\alpha \approx_W \beta$ if and only if $\varphi(\alpha/p) \in W(X)$ iff $\varphi(\beta/p) \in W(X)$
for every $\alpha, \beta, \varphi \in For, X \subseteq For$ and $p \in Var$.

A moment's reflection will show that for any operation W on 2^{For} both relations are equivalences compatible with all connectives, i.e they are congruence relations of $\mathcal{L}$.

2.1. For any consequence operation C, the relations $=_C$ and $\approx_C$ coincide.

Proof. $(\subseteq)$. Assume that C is a consequence operation and that $\alpha \approx_C \beta$. Let φ be any formula, X a set of formulas and let $p \in Var$. Then applying (T0) we get $\varphi(\beta/p)) \in C(X, \varphi(\beta/p))$ and, therefore, our second assumption implies $\varphi(\alpha/p) \in C(X, \varphi(\beta/p))$. Subsequently, using again (T0), we obtain $X \cup \varphi(\alpha/p) \subseteq C(X, \varphi(\beta/p))$ and by (T1) and (T2), we finally get $C(X, \varphi(\alpha/p)) \subseteq C(X, \varphi(\beta/p))$. The opposite inclusion is easy to get on much the same way. Therefore $\alpha =_C \beta$.

$(\supseteq)$. Assume $\alpha =_C \beta$ and take any φ, X and p. Now, if $\varphi(\alpha/p)) \in C(X)$ then, due to (T1) and (T2), $C(\varphi(\alpha/p)) \subseteq C(C(X)) = C(X)$. Since $C(\varphi(\alpha/p)) = C(\varphi(\beta/p))$, we obtain that $\varphi(\beta/p)) \in C(X)$. The symmetry is obvious and therefore $\alpha \approx_C \beta$.

In general, the W-identity and W-equivalence are related in different possible ways. They may be even independent one from the other. This option is illustrated by an example of a (structural) q-consequence operation:

2.2. Let $\mathcal{L}^* = \{For, +, -, *\}$ be a sentential language with three binary connectives. Consider the q-consequence operation $W^* = Wn_{\mathbf{R}}$ defined on $\mathcal{L}$ by the following rule of inference:

$$\mathbf{R} = \{\{\alpha\}, \beta : \alpha \in Sb(p+q) \cup Sb(p-q), \beta \in Sb(p+q) \cup Sb(p*q)\},$$

where $Sb(\alpha) = \{e\alpha : e \in End(\mathcal{L})\}$ is the set of all substitutions of the formula α. It is easy to verify that

$\alpha =_{W^*} \beta$ if and only if either $\alpha, \beta \in Sb(p+q) \cup Sb(p-q)$ or $\alpha, \beta \notin Sb(p+q) \cup Sb(p-q)$, and

$\alpha \approx_{W^*} \beta$ if and only if $\alpha, \beta \in Sb(p+q) \cup Sb(p*q)$.

Therefore, $p+q =_{W^*} p-q$ while $not(\ p+q \approx_{W^*} p-q)$, and $not(p+q =_{W^*} p*q)$ while $p+q \approx_{W^*} p*q$, what means that neither $=_{W^*} \subseteq \approx_{W^*}$ nor $\approx_{W^*} \subseteq =_{W^*}$. Thus the W^*-identity and W^*-equivalence are independent.

Two objects are identical if and only if they share exactly the same properties, cf. [7]. If we apply this *Leibniz dictum* for a q-consequence W, then $\alpha =_W \beta$ should imply for any φ, p and X, if $\varphi(\alpha/p)) \in W(X)$ then $\varphi(\beta/p)) \in W(X)$ and vice-versa. Thus, then also $\alpha \approx_W \beta$. Taking it into account a q-consequence W will be called *extensional* whenever $=_W \subseteq \approx_W$.

The property of extensionality may be naturally adopted to propositional contexts and, ultimately, to connectives. A k-argument connective f^k ($k \geq 1$) from a language in which W is defined will be called extensional for W, or W-*extensional* , if for any formulas $\alpha_1, \alpha_2, ..., \alpha_k, \beta_1, \beta_2, ..., \beta_k$:

$f^k(\alpha_1, \alpha_2, ..., \alpha_k) \approx_W f^k(\beta_1, \beta_2, ..., \beta_k)$, whenever $\alpha_1 =_W \beta_1, \alpha_2 =_W \beta_2, ..., \alpha_k =_W \beta_k$.

2.3. *Each connective of an extensional inference W is W-extensional.*

Now, one may ask when for a given q-consequence operation W the opposite inclusion holds, i.e., under which conditions $\approx_W \subseteq =_W$. A partial answer to this question may be given using the concept of expressibility of identity by a set of formulas. Where $p, q \in Var$, let $E(p, q)$ be a set of formulas, $E(p, q) \subseteq For$, built from the variables p and q. Let, subsequently, $E(\alpha, \beta)$ denote the set of formulas obtained from $E(p, q)$ after simultaneous replacement (substitution) of p, q by α and β, respectively. We shall say that $=_W$ is W-*expressible* whenever there is $E(p, q) \subseteq For$ such that for any $\alpha, \beta \in For$:

($\circ$) $\alpha =_W \beta$ if and only if $E(\alpha, \beta) \subseteq W(\emptyset)$.

In such a case one may say that the set $E(p, q)$ *expresses* the identity with respect to W. It is a routine matter to verify that

2.4. $E(p, q)$ expresses $=_W$ if and only if the following conditions are satisfied:

(1^o) $E(\alpha, \alpha) \subseteq W(\emptyset)$
(2^o) $E(\alpha, \beta) \subseteq W(\emptyset)$ if and only if $E(\beta, \alpha) \subseteq W(\emptyset)$
(3^o) If $E(\alpha, \beta) \subseteq W(\emptyset)$ and $E(\beta, \gamma) \subseteq W(\emptyset)$, then $E(\alpha, \gamma) \subseteq (\emptyset)$
(4^o) For every k-ary connective $F_i (i = 1, \ldots, m), E(\alpha_1, \beta_1) \subseteq W(\emptyset)$, $\ldots, E(\alpha_k, \beta_k) \subseteq W(\emptyset)$ imply that $E(F_i(\alpha_1, \ldots, \alpha_k), F_i(\beta_1, \ldots, \beta_k)) \subseteq W(\emptyset)$

Taking this into account, we get that

2.5. For any W-expressible identity $=_W, \approx_W \subseteq =_W$.

Proof. Assume that $E(p, q)$ expresses the relation $=_W$ and that $\alpha \approx_W \beta$. Now, consider an arbitrary $\varphi(p, q) \in E(p, q)$ and take $\varphi(\alpha/p, q) = \varphi(\alpha, q)$. Then, due to the assumption and ($\circ$)

$$\varphi(\alpha, \alpha) \in W(\emptyset) \text{ if and only if } \varphi(\alpha, \beta) \in W(\emptyset)$$

and, according to $2.4.1^0$, $\varphi(\alpha, \beta) \in W(\emptyset)$. Since φ was arbitrary, we conclude that $E(\alpha, \beta) \subseteq W(\emptyset)$ and thus, by ($\circ$), that $\alpha =_W \beta$.

Accordingly, under the assumptions of 2.5, W-equivalent formulas are W-identical. It is noteworthy that the W-expressible identity relative to an abstract inference W resembles in some way the concept of equivalence from the theory of structural consequence operation, cf. [6]. The analysis of relations between two concepts may be then an appropriate preliminary investigation of the class of q-consequences W with W-expressible identity.

3. Structurality and inferential extensions

Similarly as in the theory of consequence operation, the structurality plays an important role in the inferential approach. A q-consequence W of $\mathcal{L}$ is structural if for every *substitution* of $\mathcal{L}$ (i.e. endomorphism of $\mathcal{L}$)

(S) $eW(X) \subseteq W(eX)$.

Then, as in the standard environment, any pair $(\mathcal{L}\ , W)$ consisting of a sentential language $\mathcal{L}$ and a structural q-consequence operation W on $\mathcal{L}$ is called an *inferential logic*.

The studies in [5] show how the well known Lindenbaum-Wójcicki completeness result for structural logics $(\mathcal{L}\ , C)$ shifts onto the inferential case. To this aim, however, the notion of logical matrix had to be extended. Now, when $\mathcal{L}$ is a sentential language and $\mathcal{A}$ is an algebra similar to $\mathcal{L}$, a *q-matrix* is a triple

$$M^* = (\mathcal{A}, D^*, D),$$

where D^* and D are disjoint subsets of the universe A of $\mathcal{A}$ ($D^* \cap D = \emptyset$). D and D^* are interpreted as sets of *rejected* and *distinguished* elements (values) of M, respectively. For any such M^* one defines the relation $\models_{M*}$ between sets of formulae and formulae, a *matrix q-consequence of* M^* putting for any $X \subseteq For, \alpha \in For$

$X \models_{M^*} \alpha$ if and only if for every $h \in Hom(\mathcal{L}, \mathcal{A})(h\alpha \in D$ whenever $hX \cap D^* = \emptyset)$.

The relation of q-consequence was designed as a formal counterpart of reasoning admitting rules of inference which from non-rejected assumptions lead to accepted conclusions. With every $\models_{M^*}$ there is uniquely associated an operation $Wn_{M^*} : 2^{For} \to 2^{For}$ such that

$$\alpha \in Wn_{M^*}(X) \text{ if and only if } X \models_{M^*} \alpha.$$

The q-concepts coincide with usual concepts of matrix and consequence only if $D^* \cup D = A$, i.e. when the sets D^* and D are complementary. Then, the set of rejected elements coincides with the set of non-designated elements and the structure of matrix reduces to

$M = (A, D)$

and the q-consequence relation $\models_{M^*}$and operation Wn_{M^*} reduce to $\models_M$and Cn_M , which are defined by the formulas:

$X \models_M \alpha$ if and only if for every $h \in Hom(L, A)(h\alpha \in D$ whenever $hX \subseteq D)$, and

$$\alpha \in Cn_M(X) \text{ if and only if } X \models_M \alpha.$$

Given a matrix $M = (A, D)$ and a q-matrix $M^* = (\mathcal{A}, D^*, D)$ for a language $\mathcal{L}$, a system of sentential logic (a set of "tautologies") may be then defined as the set of all formulas taking for every valuation h (a homomorphism) of $\mathcal{L}$ in $\mathcal{A}$. Traditionally, that set is called content of the matrix M and denoted as $E(M)$. Thus,

$E(M) = \{\alpha \in For : \text{for every } h \in \text{Hom}(L, A), h(\alpha) \in D\}$.

Note, that $Wn_{M^*}(\emptyset) = Cn_M(\emptyset) = E(M)$. So,

3.1. The matrix M and the q-matrix M^*, define the same content, i.e unique system of sentential logic.

Now, recall that, according to the Lindenbaum result, every logical system may be represented as the content of some matrix, cf. [10]. Accordingly, 3.1. implies that any logical system may equally well be extended to a sentential calculus $(\mathcal{L}, C)$, where C is a structural consequence operation, or to an inferential logic $(\mathcal{L}, W)$ with W being a structural q-consequence operation. In most cases the inferential extensions are concurrent with sentential extensions, though in special circumstances only inferential extensions are sound. The method of inferential extensions have already found applications, e.g. in the construction of inferential paraconsistent version of the three-valued Łukasiewicz logic:.

3.2.(cf. [4) The three-valued Łukasiewicz matrix for the language $\mathcal{L}_k = (For, \neg, \to, \vee, \wedge, \leftrightarrow)$ takes the form

$$M_3 = (\mathcal{A}_3, \{1\}), where\ \mathcal{A}_3 = (\{0, \tfrac{1}{2}, 1\}, \neg, \to, \vee, \wedge, \leftrightarrow\},$$

and functions of $\mathcal{A}_3$ are given by the following tables:

x	¬ x
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\leftrightarrow$	0	$\frac{1}{2}$	1
0	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

The consequence determined by M_3 is such that

$$X \models_{M_3} \alpha \text{ iff for every } h \in Hom(\mathcal{L}_k, A_3)(\text{if } hX \subseteq \{1\}, \text{ then } h\alpha = 1)..$$

Accordingly, the consequence $\models_{M_3}$is explosive: $\{\alpha\ , \neg\alpha\} \models_{M_3} \beta$, for any α and β. This means that the there-valued sentential calculus of Łukasiewicz $(\mathcal{L}_k,\ Cn_{M_3})$ is not paraconsistent.

Now, retaining 1 as the accepted value and taking 0 as the only rejected element consider the following Łukasiewicz q-matrix

$$M_3^* = (A_3, \{0\}, \{1\})$$

The q-consequence determined by M_3^* is such that

$$X \models_{M_3^*} \alpha \text{ iff for every } h \in Hom(\mathcal{L}, A_3)(\text{if } hX \subseteq \{\tfrac{1}{2}, 1\}, \text{ then } h\alpha = 1),$$

if all premises are not rejected, i.e. not false, then the conclusion is accepted, i.e. true. It is easy to note that $\models_{M^*}$is paraconsistent. This holds true since there are α, β such that not $\{\alpha\ ,\ \neg\alpha\} \models_M \beta$. To see this, one may simply take $\alpha = p$ and $\beta = q$ and conclude that any valuation sending both variables p, q into $\frac{1}{2}$ falsifies the inference. Obviously, this also means that (MP) is not a rule of $\models_{M_3^*}$

4. Inferential extensions of Łukasiewicz modal logic

Łukasiewicz [2], [3] constructed a system of four-valued propositional logic, called **Ł**-modal system, in order to capture the notion of possibility. Leaving aside the evaluation of the philosophical import of the construction, which met severe criticism of contemporary modal logicians, we remark that even those who criticized the compatibility of Łukasiewicz proposal, appreciated its algebraic potential, see e.g. [1], [8]. And this is mostly why it has been chosen for investigation.

The algebra of the logic is a product of two two-element Boolean algebras with implication, negation and one-argument operations of: *assertion* A (the first) and *verum* V (the second); i.e., $(\{0,1\}, \rightarrow, \neg, A)$ and $(\{0,1\}, \rightarrow, \neg, V)$, where $A(0) = 0, A(1) = 1$, and $V(0) = V(1) = 1$. Its values are the ordered pairs (1,1), (10), (0,1), (0,0), and the operations of implication ($\rightarrow$) and (negation($\neg$) are natural compounds of their counterparts on the axes. Further to this, the possibility Δ , is identified with the cross product of A and V. Łukasiewicz also considers the twin possibility ∇, which is made when the two arguments, assertion and verum, change their places in the product. In the sequel we adopt the original simplified notations taking 1 to stand for (1,1), 2 for (1,0), 3 for (0,1) and 4 for (0,0).

The Łukasiewicz twin possibility logic algebra has the form:

$$Ł = (\{1, 2, 3, 4\}, \rightarrow, \neg, \Delta, \nabla),$$

with operations defined by the following tables:

$\to$	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

x	$\neg$ x
1	4
2	3
3	2
4	1

x	$\triangle$ x
1	1
2	1
3	3
4	3

x	∇ x
1	1
2	2
3	1
4	2

The system **Ł** of modal logic is defined on the language $\mathcal{L} = (For, \to, \neg, \Delta, \nabla)$ as the set of all formulas taking for every valuation h (i.e., a homorphism) of $\mathcal{L}$ in Ł the *distinguished value* 1, thus

$Ł = \{\alpha \in For : \text{for every } h \in Hom(\mathcal{L}, Ł), h(\alpha) = 1\}$.

4.1. (cf. [2]). The following formulas are in Ł:

(i) $\Delta(p \to \Delta p)$, $\Delta(\Delta p \to p)$ (i') $\nabla(p \to \nabla p)$, $\nabla(\nabla p \to p)$
(ii) $p \to p$, $p \to \Delta p$ (ii') $p \to p$, $p \to \nabla p$.

4.2. (cf. [2]). The formulas $\Delta p \to p$ and $\nabla p \to p$ are not in Ł.

Let us now consider the following two q-matrices related to the Łukasiewicz system Ł:

(3) $MŁ_{\Delta} = (Ł, \{3, 4\}, \{1\})$,
(4) $MŁ_{\nabla} = (Ł, \{2, 4\}, \{1\})$.

The choice of the sets of rejected and accepted elements in $MŁ_{\Delta}$ and in $MŁ_{\nabla}$ and the whole idea of considering inferential extensions of the system of modal logic are in a way connected with Łukasiewicz attempts to discern the two operators of possibility. Note, that in the first case rejected are those elements of the algebra of values which Δ "sends to" not designated values (i.e., different from 1). The q-matrices $MŁ_{\Delta}$ and $MŁ_{\nabla}$ define two inferential extensions of Ł-modal logic, i.e. the following inferential calculi:

(5) $(Ł, W_{\Delta})$ and $(\mathcal{L}, W_{\nabla})$;

we put W_{Δ} for $Wn_{MŁ_{\Delta}}$, and W_{∇} for $Wn_{MŁ_{\nabla}}$. Obviously, $W_{\Delta}(\emptyset) = W_{\nabla}(\emptyset) = Ł$.

The questions we are now going to ask will concern the characterisation of q-consequences W_{Δ}, W_{∇} and the connectives of the two logics from the point of view of extensionality and intensionality defined in Section 4. To start with let us take the set of formulas $E_1(p, q) = \{\Delta(p \to q), \Delta(q \to p)\}$. We then get

4.3. $\alpha = W_\Delta \beta$ if and only if $E_1(\alpha, \beta) \subseteq W_\Delta(\emptyset)$.

Proof. Only the direction from right to left is not trivial. So assume that $E_1(\alpha, \beta) \subseteq W_\Delta(\emptyset)$. First, one may verify that the formulas in $E_1(\alpha, \beta)$ under a valuation in L both take the value 1 whenever α and β are sent either both into $\{1, 2\}$ or into $\{3, 4\}$. In turn, an easy induction will show that for any $h \in Hom(L, \text{Ł})$

(*) $h\alpha \in \{1, 2\}$ if and only if $h\beta \in \{1, 2\}$

is equivalent to

(**) $h\varphi(\alpha/p)) \in \{1, 2\}$ if and only if $h\varphi(\beta/p)) \in \{1, 2\}$.

Therefore, for any X, φ and $p, W_\Delta(X, \varphi(\alpha/p)) = W_\Delta(X, \varphi(\beta/p))$ and thus $\alpha =_{W_\Delta} \beta$.

Subsequently, let us take $E_1(p, q) = \{p \to q, q \to p\}$. By a simple table inspection one may also show that

4.4. $\alpha \approx_{W_\Delta} \beta$ if and only if $E(\alpha, \beta) \subseteq W_\Delta(\emptyset)$.

Now observe that 4.1 (i) implies that $E_1(p, \Delta p) \subseteq W_\Delta(\emptyset)$ and, by 4.3, we finally obtain $p =_{W_\Delta} \Delta p$. Since, however, due to 4.2 $\Delta p \to p \notin W_\Delta(\emptyset)$, 4.4 yields that p and Δp are not W_Δ-equivalent, i.e., not $p \approx_{W_\Delta} \Delta p$. Therefore

4.5. W_Δ is an intensional q-consequence.

On the side of the connectives we obtain that

4.6. Δ is an extensional connective of W_Δ and $\to, \neg, \nabla$ are W_Δ-intensional.

Proof. From the proof of 4.3 it easily follows that $\alpha =_{W_\Delta} \beta$ implies that for any $h \in Hom(\mathcal{L}, \text{Ł}), h\alpha$ and $h\beta$ both are either in $\{1, 2\}$ or in $\{3, 4\}$. Thus, due to the table for $\Delta, h(\Delta\alpha) = h(\Delta\beta)$. So, $\Delta\alpha \in W_\Delta(X)$ iff $\Delta\beta \in W_\Delta(X)$ and α, β are (δ, W_Δ)-equivalent, $\alpha\ {}_\Delta \approx_{W_\Delta} \beta$. Therefore, Δ is W_Δ-extensional.

The intensionality of the remaining connectives is exemplified through examples:

($\to$) It suffices to consider the formulas used in the proof of intensionality of W_Δ, i.e., p and $\Delta p.p =_{W_\Delta} \Delta p$, but $\Delta p \to p(p/p) =\notin W_\Delta(\emptyset)$ and $\Delta p \to p(\Delta p/p) \in W_\Delta(\emptyset)$; cf. 4.1 (ii) and 4.2. Therefore, p and Δp are not W_Δ-equivalent.

($\neg$) Take $\alpha = \neg(p \to p), \beta = \neg(\Delta p \to p)$ and $\varphi = \neg p$. Using once again the property from the proof of 5.1 verify that $\Delta =_{W_\Delta} \beta$. Then, $h(\varphi(\alpha/p)) = h(p \to p)$ and $h(\varphi(\beta/p)) = h(\Delta p \to p)$ for every h. Since $p \to p \in W_\Delta(\emptyset)$ and $\Delta p \to p \notin W_\Delta(\emptyset), \alpha$ and β are not (φ, W_Δ)-equivalent.

(∇) Again as in ($\to$) we may use $p =_{W_\nabla} \Delta p$. Now, we take $\varphi = \nabla p.\varphi p/p)) = \nabla p$ and $\varphi\pi/p)) = \nabla\Delta p$. One may easily verify that $\nabla p \notin W_\Delta(\emptyset)$ while $\nabla\Delta p \in W_\Delta(\emptyset)$. So, p and Δp are not W_Δ-equivalent.

Anyone who has carefully passed through this Section will not be surprised that the characterisation of W_Δ is similar. ∇ mirrors the properties of Δ and, consequently,

4.7. W_∇ is an intensional q-consequence.

4.8. ∇ is an extensional connective of W_∇ and $\to, \neg, \Delta$ are W_∇-intensional.

5. Final remarks

The distinction between the extensionality and intensionality on the inferential level, i.e. in reference to the concept of q-consequence, is important for the inferential framework. Actually, it enables a better insight into external properties of inferential calculi, including extensions of several logical systems.

The Łukasiewicz four-valued modal matrix proved to be a good experimental range for the study of inferential extensionality and intensionality of inferential logics and modal connectives. Ł $= (\{1, 2, 3, 4\}, \to, \neg, \Delta, \nabla)$ attracted our attention not by its modal properties, which were often justly contested. Rather, it was an importance of the algebraic potential of an unusual Boolean product matrix of **Ł** modal logic. The latter has already been noticed and estimated by several scholars like Simons [8] and, more recently, Font, Hajek [1].

It is interesting that in the *Boolean* Lindenbaum algebras for W_Δ and W_∇ the two Łukasiewicz connectives of possibility identify the "inference" filters consisting of formulas, which preceded by Δ or by ∇ belong either to $W_\Delta(X)$ or to $W_\nabla(X)$, respectively. Then, the Lindenbaum q-matrices for $(\mathcal{L}, W_\Delta)$ based on X $(X \subseteq For)$ have formulas, which belong to the F_Δ-filter $W_\Delta(X)$ as "accepted", and the classes of formulas corresponding to $For - (X \cup W_\Delta(X))$ as "rejected". The situation for $(\mathcal{L}, W_\nabla)$ is, obviously, analogous. These procedures reflect direct Δ and ∇ filtrations on the generic set { 1, 2, 3, 4 }:

$F_\Delta = \{x : \Delta x = 1\} = \{\ 1,\ 2\ \}$,
$F_\nabla = \{x : \nabla x = 1\} = \{\ 1,\ 3\ \}$.

It seems that algebraic investigation of congruences of the Boolean **Ł**-models having more elements may result in a better characterisation of both modal connectives. In a wider perspective, such investigation would be a good starting point for getting other interesting results in the theory of inferential logics.

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