A Decidability Proof for Propositional Neighborhood Logic (Extended Abstract)

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In memoriam A. Mostowski, H. Rasiowa and C. Rauszer

1 Introduction

Interval-based temporal reasoning over partially/linearly ordered domains is widely recognized as an important research area in various fields of computer science and artificial intelligence. Unfortunately, interval-based temporal logics often present a bad computational behavior. As for the propositional setting and the linearly ordered domains, the most significant interval temporal logics are Halpern and Shoham's Modal Logic of Time Intervals (HS) [6], which has been shown to be undecidable over several classes of linear orderings by a reduction from the halting problem, Venema's CDT [16], which is expressive enough to embed the whole HS and thus it is undecidable at least on the classes of linear orderings on which HS is undecidable, the *begins/ends* fragment of HS (BE), which has been shown to be undecidable over dense [7] and discrete [4] linear orderings, and Moszkowski's Propositional Interval Logic PITL [12], that can be viewed as a fragment of CDT, which is undecidable over discrete and dense linear orderings. A comprehensive and up-to-date survey on the main developments, results, and open problems in the area of propositional interval temporal logics can be found in [4].

In this paper we focus our attention on Propositional Neighborhood Logic (PNL). PNL is a (proper) fragment of HS featuring two modalities only, which corresponds to Allen's relations *meets* and *met by*. A sound and complete axiomatic systems for PNL and a tableau-based semi-decision procedure for it have been developed in [3]. A tableau-based decision procedure for the future fragment of PNL (RPNL), over discrete linear orderings, has been recently proposed in [1]. However, such a procedure cannot be easily generalized to the case of full PNL (as a matter of fact, the procedure outlined in [9] does not work properly). The main contribution of the present paper is a proof of the decidability of the satisfiability problem for full PNL over different classes of linear orderings by a reduction to the two-variable fragment of first-order logic over ordered domains. To the best of our knowledge, apart from the case of RPNL, this is the first non-trivial case of a decidable propositional interval logic interpreted over *fully-instantiated* temporal structures, that is, temporal structures containing all intervals that can be built up from a given linear ordering over points, which does not resort to any projection principle, e.g., locality or homogeneity [6]. We conclude the paper with a discussion of decidability issues for interval temporal logics and a short comparison between PNL and point-based temporal logics, that allows us to point out the expressive strength of PNL.

2 Propositional Neighborhood Logic (PNL)

The language of the *Propositional Neighborhood Logic* (PNL for short) consists of a set of propositional variables $\mathcal{AP}$, the Boolean connectives $\neg$ and $\vee$, and the modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$, with the dual modalities $[A]$ and $[\overline{A}]$ (we assume here the so-called *strict* semantics, that is, we do not allow intervals with coincident endpoints; however, all results can be generalized to the *non-strict* case). The other classical Boolean connectives can be considered as abbreviations. The *formulas* of PNL^-, denoted by $\phi, \psi, \ldots$, are recursively defined as follows:

$$\phi := p \mid \neg\phi \mid \phi \vee \psi \mid \langle A\rangle\phi \mid \langle\overline{A}\rangle\phi.$$

The semantics of PNL^- is given in terms of *models* of the form $\mathbf{M} = \langle \mathbb{D}, V\rangle$, where the pair $\mathbb{D} = \langle D, <\rangle$ is a linearly ordered set and V is a *valuation function* for the propositional variables. Given a linear ordering $\mathbb{D} = \langle D, <\rangle$, the set of all strict intervals, called *interval structure*, is denoted by $\mathbb{I}(\mathbb{D})$. The valuation function is a mapping $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(D)}$ in such a way that, for any $p \in \mathcal{AP}$, $[d_0, d_1] \in V(p)$ if (and only if) p is *true* over $[d_0, d_1]$. The *truth* relation at a given interval in a model $\mathbf{M}$ is defined by induction on the structural complexity of formulas:

(1) $\mathbf{M}, [d_0, d_1] \Vdash p$ iff $[d_0, d_1] \in V(p)$, for all $p \in \mathcal{AP}$;
(2) $\mathbf{M}, [d_0, d_1] \Vdash \neg\psi$ iff it is not the case that $\mathbf{M}, [d_0, d_1] \Vdash \psi$;
(3) $\mathbf{M}, [d_0, d_1] \Vdash \phi \vee \psi$ iff $\mathbf{M}, [d_0, d_1] \Vdash \phi$ or $\mathbf{M}, [d_0, d_1] \Vdash \psi$;
(4) $\mathbf{M}, [d_0, d_1] \Vdash \langle A\rangle\psi$ iff there exists d_2 s.t. $d_1 < d_2$ and $\mathbf{M}, [d_1, d_2] \Vdash \psi$;
(5) $\mathbf{M}, [d_0, d_1] \Vdash \langle\overline{A}\rangle\psi$ iff there exists d_2 s.t. $d_2 < d_0$ and $\mathbf{M}, [d_2, d_0] \Vdash \psi$.

PNL is powerful enough to express interesting temporal properties [3]. Beside the ability of characterizing various properties of the underlying linear ordering, the language of PNL allows one to express the *difference* operator, and consequently to simulate *nominals*. Therefore, every universal property of strict interval structures can be expressed in such a language.

3 Decidability of PNL

We prove the decidability of PNL by embedding it into 2FO[<]. 2FO[<] is the fragment of first-order logic (with equality) over ordered domains, whose language includes a binary relation $<$ interpreted as the linear ordering relation and uses only two distinct (possibly reused) variables. 2FO[<] formulas are denoted here by $f, g, \ldots$. Notice that we do not lose generality by restricting ourselves to unary and binary predicates only; indeed, it is possible to show that in a language with at most two variables, relational symbols of any arity greater than two can be discarded [5]. Models of 2FO[<]-formulas are classical first-order interpretations I where the symbol $<$ is interpreted as a linear ordering relation. The decidability of the satisfiability problem for the 2FO fragment, without equality, has been shown by Scott in [14]. Later, Mortimer extended this result to 2FO with equality [11]. More recently, Grädel, Kolaitis and Vardi improved Mortimer's result by lowering the complexity bound [5]. Finally, in [13] Otto showed that the satisfiability problem for the two-variable first-order logic interpreted over a linear ordering 2FO[<] can be decided in CO-NEXPTIME.

PNL can be translated into 2FO[<] as follows. Let us assume that for every propositional variable $p \in \mathcal{AP}$, there exists a corresponding binary relation $p(x, y)$ in 2FO[<]. The translation function $ST_{x,y}$, where x, y are two (free) first-order variables, is defined by the following rules:

- $ST_{x,y}(p) = p(x, y) \wedge x < y$;
- $ST_{x,y}(\neg\phi) = x < y \wedge \neg ST_{x,y}(\phi)$;
- $ST_{x,y}(\phi \vee \psi) = x < y \wedge (ST_{x,y}(\phi) \vee ST_{x,y}(\psi))$;
- $ST_{x,y}(\langle A\rangle\phi) = x < y \wedge \exists x ST_{y,x}(\phi)$;
- $ST_{x,y}(\langle \overline{A}\rangle\phi) = x < y \wedge \exists y ST_{y,x}(\phi)$.

As we shall later point out, two variables are not sufficient to translate other more expressive interval temporal logics, such as, for instance, HS and CDT.

Theorem 1 *Given a* PNL*-formula ϕ, we have that ϕ is satisfiable if and only if $ST_{x,y}(\phi)$ is satisfiable.*

Proof.

We preliminary introduce a bijective mapping between interval models and first-order interpretations for 2FO[<]. Let $\mathbf{M} = \langle \mathbb{D}, V\rangle$ be an interval model. The corresponding first-order interpretation $I = \eta(\mathbf{M})$ is defined as follows: (i) the domain $\langle D', <'\rangle$ of I is any linearly ordered set such that there is an isomorphism $\eta : \langle D, <\rangle \mapsto \langle D', <'\rangle$; (ii) the valuation of the relational symbols $p(x, y)$ in the language of 2FO[<] is such that $p(d'_i, d'_j)$ holds if and only if $p \in V([d_i, d_j])$, where $\eta(d_i) = d'_i$ and $\eta(d_j) = d'_j$. For any PNL-formula ϕ, we prove that $\mathbf{M}, [d_0, d_1] \Vdash \phi$ if and only if $\eta(\mathbf{M}), [x/\eta(d_0), y/\eta(d_1)] \models ST_{x,y}(\phi)$. The proof is by induction on the structural complexity of ϕ. The base case, as well as the case of Boolean connectives, are straightforward, and thus omitted. Let $\phi = \langle A\rangle\psi$, and suppose that, for a given model $\mathbf{M}$ and interval $[d_0, d_1]$, $\mathbf{M}, [d_0, d_1] \Vdash \phi$. By the semantics of PNL there exists an element d_2 such that $d_1 < d_2$ and $\mathbf{M}, [d_1, d_2] \Vdash \psi$. By the definition of $ST_{x,y}$, we have that $ST_{x,y}(\langle A\rangle\psi) = x < y \wedge \exists x ST_{y,x}(\psi)$. By definition of η, there exist three points $d'_0, d'_1, d'_2 \in D'$ such that $d'_0 = \eta(d_0)$, $d'_1 = \eta(d_1)$, $d'_2 = \eta(d_2)$, and $d'_0 <' d'_1 <' d'_2$. By inductive hypothesis, we have that $\mathbf{M}, [d_1, d_2] \Vdash \psi$ is true if and only if

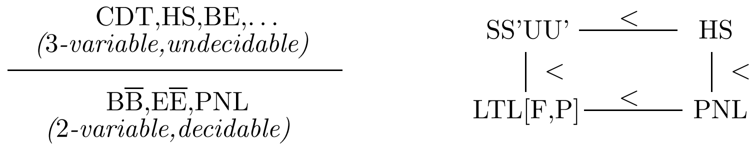

Figure 1: 2 vs. 3-variable characterization of interval logics (left), and interval vs point-based logics (right).

$\eta(\mathbf{M}), [y/\eta(d_1), x/\eta(d_2)] \models ST_{y,x}(\psi)$. It immediately follows that $\mathbf{M}, [d_0, d_1] \Vdash \langle A \rangle \psi$ if and only if $\eta(\mathbf{M}), [x/\eta(d_0), y/\eta(d_1)] \models x < y \wedge \exists x ST_{y,x}(\psi)$. The case $\phi = \langle \overline{A} \rangle \psi$ can be dealt with in a very similar way, and thus it is omitted. ∎

Since the above translation is polynomial in the size of the input formula, we have that the satisfiability problem for PNL, interpreted over the class of all strict interval structures, is decidable in CO-NEXPTIME. Moreover, since PNL is expressive enough to define discrete, dense, Dedekind complete, bounded, and unbounded linear orderings [3], such a decidability result holds for these specific classes of structures as well.

4 On the decidability of interval temporal logics

The problem of finding decidable interval temporal logics has been raised by several authors, e.g., [6, 16]. As a matter of fact, in interval temporal logic literature undecidability is the rule and decidability the exception. Interval logics make it possible to express properties of *pairs* of time points, rather than *single* time points and, in most cases, this feature prevents one from the possibility of reducing interval-based temporal logics to point-based ones. However, there are a few exceptions where the logic satisfies suitable *syntactic and/or semantic restrictions*, and such a reduction can be defined, thus allowing one to benefit from the good computational properties of point-based logics [8]. More precisely, decidability has been achieved (i) by restricting the set of modalities, (ii) by assuming suitable projection principles at the semantic level, or (iii) by interpreting the logics over non fully-instantiated temporal structures. The first approach has been successfully applied to the B$\overline{\text{B}}$ fragment of HS (as well as to its past counterpart E$\overline{\text{E}}$). B$\overline{\text{B}}$ is the fragment of HS that only features the *begins* ($\langle B \rangle$) and *begun by* ($\langle \overline{B} \rangle$) modalities. The decidability of B$\overline{\text{B}}$ can be obtained by embedding it into the point-based temporal logic of linear time (LTL[F,P]) with temporal modalities F (sometime in the future) and P (sometime in the past) [4]. The second approach has been followed by Moszkowski in [12]. He tailored a decidable fragment of PITL extended with quantification over propositional variables (QPITL) by imposing a *locality* constraint stating that a propositional variable is true over an interval if and only if it is true at its starting point. By exploiting such a constraint, decidability of QPITL can be proved by embedding it into quantified LTL. The latter approach achieves decidability by constraining the classes of temporal structures over which the interval logic is interpreted. This is the case with the so-called Split Logics (SLs) investigated by Montanari et al. in [10]. SLs are propositional interval logics equipped with operators borrowed from HS and CDT, but interpreted over specific structures, called *split structures*. The distinctive feature of split structures is that every interval can be 'chopped' in at most one way. The decidability of various SLs has been proved by embedding them into the first-order fragments of monadic second-order decidable theories of time granularity. The decidability result for PNL given in this paper suggests a different classification of interval temporal logics based on the number of variables needed to the translate them in first-order logic: we have that PNL, $\mathrm{B}\overline{\mathrm{B}}$, and $\mathrm{E}\overline{\mathrm{E}}$ are two-variable and thus decidable, while, using Ehrenfeucht-Fraïssè games, it can be shown that the undecidable logics HS, CDT, and BE are three-variable (see Figure 1, left).

5 PNL and point-based temporal logics

It is possible to compare the expressive power of PNL with that of point-based temporal logics interpreted over the class of all linear orderings. In [15], Venema shows that HS (and thus CDT as well) is strictly more expressive than any point-based temporal logic over the class of all linear orderings (by comparing HS with SS'UU', which is the most expressive point-based temporal logic). We give a similar result for PNL (see Figure 1, right): PNL, which is strictly less expressive than HS, is strictly more expressive than LTL[F,P]. First, it is possible to show that PNL is at least as expressive as LTL[F,P] by providing a satisfiability-preserving polynomial translation of LTL[F,P]-formulas into PNL. According to this translation, a propositional variable holds at a time point d_0 of a LTL[F,P]-model if and only if it holds over all intervals ending in d_0 of the corresponding PNL model. To prove that PNL is strictly more expressive than LTL[F,P], we show that there exists at least one property of the underlying frame (in the case of all linear orderings) that PNL is able to express, while LTL[F,P], interpreted over the same class of linear orderings, is not. This property is the following one: *every point which has a successor has an immediate one* (Lemma 9.13, page 351). In [2] Gabbay et al. have shown that LTL[F,P] cannot express it, while Goranko et al. have shown that it can be expressed in PNL [3].

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