ON THE REPRESENTATIONS OF DeMORGAN ALGEBRAS

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Let Q be an arbitrary non-empty set, let $O_p^{(n)}Q$ be a set of all n-ary operations on Q, and

$$O_pQ = \bigcup_n O_p^{(n)}Q;$$

For every non-empty subset $\Sigma \subseteq O_pQ$, the pair $(Q; \Sigma)$ is called an algebra.

A bisemigroup is an algebra $Q(\cdot, \circ)$ equipped with two binary associative operations $\cdot$ and $\circ$. If both of these operations have an identity element, then the bisemigroup is called a bimonoid. A commutative bisemigroup is a bisemigroup in which both operations are commutative. A bisemilattice is a commutative bisemigroup in which both operations are idempotent. In any bisemilattice $Q(\cdot, \circ)$, binary operations determine two partial orders $\leqslant_1$ and $\leqslant_2$. A bisemilattice is called a bilattice, if the partial orders $\leqslant_1$ and $\leqslant_2$ are lattice orders. Since every lattice order is characterized by two binary operations, every bilattice is a binary algebra with four operations and corresponding identities. A DeMorgan bisemigroup ([Brzo;00]) is an algebra $Q(\cdot, \circ, ^-, 0, 1)$ such that $Q(\cdot, \circ)$ is a bimonoid with identity elements 0 (for operation $\cdot$), 1 (for operation $\circ$) and such that identities

$$\overline{\overline{x}} = x,$$
$$\overline{x \cdot y} = \overline{x} \circ \overline{y},$$
$$\overline{x \circ y} = \overline{x} \cdot \overline{y},$$
$$x \circ 0 = 0 \circ x = 0,$$
$$x \cdot 1 = 1 \cdot x = 1$$

hold. A DeMorgan bisemigroup $Q(\cdot, \circ, ^-, 0, 1)$ is called a DeMorgan algebra ([Moi;35], [Mark;50], [Bia-Ras;57], [Kal;58], [San;78]), if $Q(\cdot, \circ)$ is a distributive lattice. The Lukasiewicz 3-valued logic $\mathbf{L_3}$ and the Kleene's 3-valued logic $\mathbf{K_3}$ are DeMorgan algebras.

Every two element DeMorgan algebra is a Boolean algebra. We shall call an element a of a given DeMorgan algebra a fixed point if $\overline{a} = a$. A DeMorgan algebra with four elements and two fixed points is uniquely determined up to isomorphism and will be denoted by **4**. A DeMorgan algebra with three elements is uniquely determined up to isomorphism and will be denoted by **3**.

In 1957, Bialynicki-Birula and Rasiowa characterized these lattices under the name quasi-Boolean algebras ([Bia-Ras;57]). They showed that every quasi-Boolean algebra is isomorphic to a quasi-field of sets. On the characterization of DeMorgan algebras may be seen also in [Brzo;01].

Let $\mathfrak{L}$ be a bounded distributive lattice, $\mathfrak{L}^{op}$ be the dual bounded distributive lattice of $\mathfrak{L}$ and $\mathfrak{L} \times \mathfrak{L}^{op}$ be their direct product. By equality

$$\overline{(x,y)} = (y,x)$$

we convert the bounded distributive lattice $\mathfrak{L} \times \mathfrak{L}^{op}$ into a DeMorgan algebra. Every subalgebra of DeMorgan algebra $\mathfrak{L} \times \mathfrak{L}^{op}$ is also a DeMorgan algebra.

Theorem 1.1 *Every DeMorgan algebra is isomorphic to a subalgebra of* $\mathfrak{L} \times \mathfrak{L}^{op}$ *for some bounded distributive lattice* $\mathfrak{L}$.

The next result is a characterization of DeMorgan algebras by fuzzy sets ([Gog;67]). Let X be a non-empty set, let $L(+,\cdot,^{-},0,1)$ be a DeMorgan algebra and let L^X be the set of all mappings $X \to L$. The set L^X converts to a DeMorgan algebra under the following operations:

$$\begin{aligned} (f \vee g)x &= f(x) + g(x), \\ (f \wedge g)x &= f(x) \cdot g(x), \\ \overline{(f)}x &= \quad \overline{f(x)}. \end{aligned}$$

Theorem 1.2 *Every DeMorgan algebra is isomorphic to a subalgebra of* $\mathbf{4}^X$ *for some set* X.

An algebra $L(+,\cdot,^{-},0,1,a)$ with two binary, one unary and three nullary operations is called a Kleene algebra, if $L(+,\cdot,^{-},0,1)$ is a DeMorgan algebra satisfying $\overline{a} = a$ and $x + \overline{x} + a = x + \overline{x}$. Every Kleene algebra has only one fixed point a. DeMorgan algebra $\mathbf{3}$ is a smallest Kleene algebra, hence $\mathbf{3}^X$ is a Kleene algebra as well for any set X.

Theorem 1.3 *Every Kleene algebra is isomorphic to a subalgebra of* $\mathbf{3}^X$ *for some set* X.

An algebra $Q(\cdot,\circ,^{-},',0,1)$ with two binary ,two unary and two nullary operations is called a Boolean bisemigroup, if $Q(\cdot,\circ,^{-},0,1)$ is a DeMorgan algebra, $Q(\cdot,\circ,',0,1)$ is a Boolean algebra and unary operations are commute. Every Boolean algebra is a Boolean bisemigroup with equal unary operations. DeMorgan algebra $\mathbf{4}$ is a Boolean bisemigroup as well.

Let $\mathfrak{B}$ be a Boolean algebra, $\mathfrak{B}^{op}$ be the dual Boolean algebra of $\mathfrak{B}$ and $\mathfrak{B} \times \mathfrak{B}^{op}$ be their direct product. By equality

$$\overline{(x,y)} = (y,x)$$

we convert the Boolean algebra $\mathfrak{B} \times \mathfrak{B}^{op}$ into a Boolean bisemigroup. Hence every subalgebra of Boolean bisemigroup $\mathfrak{B} \times \mathfrak{B}^{op}$ is also a Boolean bisemigroup. In particular, if $\mathfrak{B}$ is a Boolean algebra of all subsets of set I, then every subalgebra of Boolean bisemigroup $\mathfrak{B} \times \mathfrak{B}^{op}$ is called a natural Boolean bisemigroup of set I.

Theorem 1.4 *Every Boolean bisemigroup is isomorphic to a subalgebra of* $\mathfrak{B} \times \mathfrak{B}^{op}$ *for some Boolean algebra* $\mathfrak{B}$.

Theorem 1.5 *Every Boolean bisemigroup is isomorphic to a natural Boolean bisemigroup of some set I.*

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Let $\mathfrak{A} = (Q;\ \Sigma)$ be an arbitrary algebra. n-ary term operations of algebra $\mathfrak{A}$ are defined by the following induction:

1) all n-ary identical operations (or projections) of set Q

$$\delta_n^i(x_1,\ \ldots\ ,\ x_n) = x_i, \qquad i = 1,\ \ldots\ ,\ n,$$

are n-ary term operations of $\mathfrak{A}$;

2) if $f_1,\ \ldots\ ,\ f_m$ are n-ary term operations of $\mathfrak{A}$, then the superposition

$$\mu_m^n(f,\ f_1,\ \ldots\ ,\ f_m)(x_1,\ \ldots\ ,\ x_n) = f(f_1(x_1,\ \ldots\ ,\ x_n),\ \ldots\ ,\ f_m(x_1,\ \ldots\ ,\ x_n))$$

is again an n-ary term operation of $\mathfrak{A}$, for every m-ary $f \in \Sigma$.

The operation $h \in O_pQ$ is called a term operation of $\mathfrak{A} = (Q;\ \Sigma)$, if h is an n-ary term operation of $\mathfrak{A}$ for some n. For $n = 1, 2, 3$ the n-ary term operation is called unary,binary,ternary.

If we denote the set of all n-ary term operations of algebra $\mathfrak{A} = (Q;\ \Sigma)$ by $\mathcal{F}^n(\Sigma)$ and

$$\mathcal{F}(\Sigma) = \mathcal{F}^1(\Sigma) \bigcup \mathcal{F}^2(\Sigma) \bigcup \ldots$$

is the set of all its term operations, then algebra $\mathcal{F}(\mathfrak{A}) = (Q,\ \mathcal{F}(\Sigma))$ is called an algebra of term operations (functions) for $\mathfrak{A}$ (or a termal algebra for $\mathfrak{A}$).

Let's consider the following binary associative multiplications ([Mann;44]):

$$f \cdot g(x, y) = f(x, g(x, y)),$$

$$f \circ g(x, y) = f(g(x, y), y).$$

If f and g are binary term operations of any algebra, then $f(x, g(x, y))$ and $f(g(x, y), y)$ are also binary term operations, hence for every f and g binary term operations there exist binary term operations h and h' with identities:

$$f(x, g(x, y)) = h(x, y), \tag{2.1}$$

$$f(g(x, y), y) = h'(x, y). \tag{2.2}$$

So the set $\mathcal{F}^2(\Sigma)$ of all binary term operations of any algebra $\mathfrak{A} = (Q;\ \Sigma)$ is a bimonoid of operations on Q.

The equations (2.1) and (2.2) have the meaning of $\forall\exists(\forall)$-identities in termal algebra.

The bimonoid $\mathcal{F}^2(\Sigma)$ is called a bimonoid of binary term operations of algebra $\mathfrak{A} = (Q;\ \Sigma)$ (or bimonoid of algebra $\mathfrak{A}$ in short).

Besides, the dual operation of every binary term operation is also a binary term operation, i.e. for every binary term operation f there exists a binary term operation f^* with identity: $f(x, y) = f^*(y, x)$, and the mapping $^- : f \to f^*$ is an antiautomorphism of bimonoid $\mathcal{F}^2(\Sigma)$.

So the set $\mathcal{F}^2(\Sigma)$ of all binary term operations of any algebra $\mathfrak{A} = (Q;\ \Sigma)$ is a DeMorgan bisemigroup (of operations on Q) with an involution $^- : f \to f^*$, and every commutative binary term operation is a fixed point of this involution.

1) If $\mathfrak{A} = (Q;\, \Sigma)$ is a non-trivial lattice or a semilattice, then

$$\mathcal{F}^2(\Sigma) = \Sigma \bigcup \left\{\delta_2^1,\, \delta_2^2\right\}.$$

For any binary term operations f, g of lattice or semilattice the following identities (hyperidentities ([Mov;92]-[Mov;98]), [Tay;81]) are valid:

$$f(x,\, x) = x,$$
$$f(x,\, f(y,\, z)) = f(f(x,\, y), z),$$
$$f(f(x,\, y), f(u,\, v)) = f(f(x,\, u), f(y,\, v)),$$
$$f(g(f(x,\, y),\, z),\, g(y,\, z)) = g(f(x,\, y),\, z),$$
$$f(x, f(x, y)) = f(x, y),$$
$$f(f(x, y), y) = f(x, y),$$
$$f(x, g(x, y)) = g(x, f(x, y)),$$
$$f(g(x, y), y) = g(f(x, y), y),$$
$$f(x, f(g(x, y), y)) = f(x, y),$$
$$f(f(x, g(x, y)), y) = f(x, y).$$

Proposition 2.1 *The bimonoid of binary term operations of any semilattice is a lattice of order 3, hence this lattice is distributive and consequently is a Kleene algebra of order 3.*

Proposition 2.2 *The bimonoid of binary term operations of any non-trivial lattice is a Boolean bisemigroup of order 4 (with two fixed points).*

2) If $\mathfrak{A} = Q(\circ)$ is a non-commutative and idempotent semigroup, then

$$\mathcal{F}^2(\{\circ\}) = \left\{\circ,\, \delta_2^1,\, \delta_2^2,\, f_1,\, f_2,\, f_3\right\},$$

where

$f_1(x,\, y) = y \circ x,$
$f_2(x,\, y) = x \circ y \circ x,$
$f_3(x,\, y) = y \circ x \circ y.$

Theorem 2.3 *The bimonoid of binary term operations of non-commutative and idempotent semigroup is a distributive lattice of order 6, i.e. a DeMorgan algebra of order 6.*

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