The universal quantifier is not definable from the existential one in the second order intuitionistic propositional logic

Konrad Zdanowski
Institute of Mathematics
Polish Academy of Science

20th September 2005

Abstract

We consider the logic with connectives $\bot, \to, \wedge, \vee$. Negation $\neg\varphi$ is defined as a shorthand for $\varphi \to \bot$. It is know that in intuitionistic propositional logic (IPC) no connective is definable from the other ones. On the other hand, in its second order version (IPC^2) one can define $\bot, \wedge, \vee, \exists$ from $\forall$ and $\to$. In this paper we present a simple semantic argument that $\forall$ is not definable from the remaining operators. Up to our knowledge it is the first argument for this fact. Moreover, we prove that for an arbitrary topology $\mathcal{T}$ and for each sentence φ without universal quantifier, φ is true in $\mathcal{T}$ if and only if it is true classically. Let us mention, that this fact does not extend to all formulas without $\forall$. Indeed, the formula $\neg\neg p \to (\exists q(p \to (q \vee \neg q)) \to p)$ is true classically but it is not true in the topology of $\{0\} \cup \{1/(n+1) : n \in \omega\}$ where the value of p is the set $\{1/(n+1) : n \in \omega\}$.

As was observed by Połacik in [**?**], the formula $\exists p((r \to (p \vee \neg p)) \to r)$ is not equivalent to any IPC formula. It follows that $\exists$ is not definable from the propositional connectives: $\bot, \wedge, \vee, \to$. Thus, the second order intuitionistic propositional logic without universal quantifier is strictly between IPC and IPC^2.

The main results of the paper are the following.

Theorem 1 *For each sentence $\varphi \in \mathcal{F}_\exists$ and for each topology $\mathcal{T} = (X, \mathcal{O}(X))$, the sentence φ is true in $\mathcal{T}$ if and only if φ is a classical tautology.*

Theorem 2 *The universal quantifier is not definable from $\bot, \vee, \wedge, \to$, $\exists$ in the second order intuitionistic propositional logic.*

Acknowledgments I want to say the BIG THANKS to Paweł Urzyczyn. This paper emerged from the lectures on intuitionistic logic he gave at Warsaw University and from the questions he put. I profited greatly from his willingness to share his thoughts and the paper profited from the comments he made on the first version of it.